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# SCHOOL ALGEBRA

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NEW YORK ·:· CINCINNATI ·:· CHICAGO

AMERICAN BOOK COMPANY

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# PREFACE

THE **general features** of this book are based on the following considerations:

(1) *Algebra is a continuation and generalization of arithmetic.* As the student comes to see this he advances in the subject with increasing ease, interest, and thoroughness. In this book, especially in Part One, the treatment is largely arithmetical. What are usually called "algebraic expressions" and "quantities" are herein called *numbers*, and treated accordingly. In short, Part One is built upon the pupil's knowledge of arithmetic, and leads him at once, without formality or abstract manipulations, into the use of the equation for solving problems.

(2) *The transition from arithmetic to algebra should be gradual and progressive.*

The chief particulars in which algebra is an extension of arithmetic should be presented one at a time. The order in which this is accomplished in this book may be seen by reference to the table of contents.

(3) *The treatment should be both topical and spiral.*

In this book, while the treatment of each new subject is thorough as far as it goes, no attempt is made to teach all about a subject on its first presentation. As new ideas are introduced there arises a necessity, as well as a pedagogic importance, for viewing each subject a second or third time in a manner more profound than is possible on first view. In Chapter I operations, signs, equations, and problems are introduced, and treated with reference to simple positive integers. Each of

the following chapters is largely a review and extension of what has gone before.

Among the **special features** are:

(1) *The method of treating zero and equality conditions.*

In some important particulars, as will be seen, this is a wide departure from the texts of the day, yet it has not been made different for the sake of novelty. See pages 141, 142, 143, 144, 228, 229.

(2) *Simplicity and thoroughness throughout the book.*

This will be noticeable especially in the important subjects of factoring, fractions, simple and quadratic equations, variation, and the progressions.

The matter on graphs, Chapter XVI, can be introduced as early as the teacher wishes. It may be advantageously taught in connection with simple and quadratic equations involving two unknown numbers.

The problems, as will be seen, are numerous and well graded, and have been selected largely with a view to indicating the utility of algebra in physics and geometry, and in the common concerns of life.

J. W. NICHOLSON.

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# PART ONE — PRIMARY ALGEBRA

## I. POSITIVE SIMPLE INTEGERS

### INDUCTION

1. How many 2's are 2, 2 and 2, 2, 2?

2. How many $x$'s are $x$, $x$, $x$ and $x$, $x$, $x$?

3. How many $a$'s are $7a$ plus $3a$? $7a + 3a = ?$

$a$, $a$, $a$, $a$, $a$, $a$, $a$
$a$, $a$, $a$

4. How many $b$'s are $5b$ less $3b$? $5b - 3b = ?$

$\not b$, $\not b$, $\not b$, $b$, $b$
$\not b$, $\not b$, $\not b$

5. How many $c$'s are $2$ times $3c$? $3 \times 2c = ?$

$c$, $c$, $c$
$c$, $c$, $c$

6. How many $d$'s are $\frac{1}{3}$ of $12d$? $12d \div 4 = ?$

$d$, $d$, $d$, $d$
$d$, $d$, $d$, $d$

7. How many times does $12d$ contain $4d$? $12d \div 4d = ?$

$d$, $d$, $d$, $d$

Find the values of the following:

8. $12\not a + 9\not a$; $15a + 6a$; $25b + b$; $18y + 7y$.

9. $12\not a - 9\not a$; $15a - 6a$; $25b - b$; $18y - 7y$.

10. $2 \times 8\not a$; $3 \times 12a$; $20b \times 4$; $40y \times 5$.

11. $\frac{1}{2}$ of $8\not a$; $\frac{1}{3}$ of $12a$; $20b \div 4$; $40y \div 5$.

12. $10\not a \div 2\not a$; $30a \div 6a$; $54b \div 9b$; $56y \div 8y$.

The signs $+$, $-$, $\times$, $\div$ stand for the same operations in algebra as in arithmetic.

### ADDITION AND SUBTRACTION

**2. Addition** is the operation of uniting two or more numbers into one number.

The **addends** are the numbers added, and the **sum** is the result of addition.

The **sign** of addition is $+$. The name of this sign is **plus**, but it is often read *and*.

The **sign of equality** is $=$, which is read *equals*.

Read: $9x + 3x = 12x$. Point out the addends; the sum.

**3. Subtraction** is the operation of finding an addend when the sum and the other addend are given.

The **minuend** is the given sum; the **subtrahend** is the given addend; and the **difference** is the required addend.

The **sign** of subtraction is $-$. The name of this sign is **minus**, but it is often read *less*.

Thus, $13 - 8 = 5$, because $8 + 5 = 13$. Here 13 is the minuend, 8 the subtrahend, and 5 the difference.

NOTE. When we speak of the difference between two numbers, the first number named is the minuend and the other is the subtrahend.

Thus, the difference between 11 and 7 is $11 - 7$, or 4.

Find the sum and difference of:

1. 20 and 8; $9x$ and $6x$; $12a$ and $8a$; $6n$ and $n$.
2. 13 and 9; $8y$ and $3y$; $18c$ and $6c$; $9x$ and $x$.

Find the values of:

| | | | | | |
|---|---|---|---|---|---|
| 3. | $14 + 6$ | $21a - 7a$ | $30g - 9g$ | $x + x$ | $y - y$. |
| 4. | $25 - 8$ | $16c - 9c$ | $17c + 7c$ | $a + a$ | $z - z$. |
| 5. | $42 - 5$ | $19x + 6x$ | $32y - 5y$ | $n - n$ | $c + c$. |

### EXERCISES IN ADDITION AND SUBTRACTION

**4.** **1.** If $a$ stands for, or represents, 5, what number is represented by $a + 6$? $a - 3$? $8 + a$? $9 - a$?

**2.** If $b = 7$, what is the numerical value of $b + 9$? $b - 5$? $12 + b$? $13 - b$?

**3.** How much more than 8 is $8 + 3$? $8 + x$?

**4.** How much less than 9 is $9 - 5$? $9 - y$?

**5.** How much more than $c$ is $c + 6$? $c + n$?

**6.** How much less than $m$ is $m - 4$? $m - b$?

**7.** What is the sum of $a$ and 3? $a$ and $n$?

**8.** What is the difference between $c$ and 6? $a$ and $b$?

**9.** James has $x$ dollars and John has 3 dollars more than James. How many dollars has John?

**10.** Frank had $y$ apples, and gave 6 of them to his sister. How many apples had he left?

**11.** What number is 1 more than the number $n$? 2 less than the number $m$?

**12.** Ann has 5 roses and Ann and Lucy together have $c$ roses. How many roses has Lucy?

**13.** Frank paid \$3 for a hat and $x$ dollars for a gun. How much more did the gun cost than the hat?

**14.** Henry jumped 9 feet and Walter $x$ feet. How much did Walter beat Henry?

**15.** A father is $x$ years old and his son is $y$ years old. How many years older is the father than the son?

**16.** In a certain school there are $7x$ boys and $9x$ girls. How many students are there in the school?

**17.** A farmer had $15y$ sheep. After selling $8y$ sheep, how many had he left?

## MULTIPLICATION

**5.** 1. How many $x$'s are $3x + 3x + 3x + 3x$?

2. How many $x$'s are 4 times $3x$?

**Multiplication** is the operation of finding the sum of equal numbers, or the operation of taking one number as many times as there are ones in another number.

The **multiplicand** is the number taken or multiplied; the **multiplier** is the number that shows how many times the multiplicand is taken; and the **product** is the result of multiplication.

The **sign** of multiplication is $\times$, which is read "*times*" or "*multiplied by*" according as it is written after or before the multiplier.

Thus, $5 \times a$ is read "5 times $a$" or "5 multiplied by $a$," according as 5 or $a$ is considered the multiplier.

**6.** An **abstract number** is a number that is not applied to any particular thing; as 3, 10, $a$, $m$, $x$.

Quantities like \$12, $b$ yards, etc., are called **concrete numbers.**

*The multiplier is always an abstract number.*

A product like $5 \times a$ is usually written $5a$. In this product 5 and $a$ are called **factors**, 5 being the **numerical factor** and $a$ the **literal factor.**

**7.** The **coefficient** of a literal factor is the number that shows how many times the factor is taken as an addend.

Thus, in $4n$, 4 is the coefficient of $n$, because $4n$ means $4 \times n$ or $n + n + n + n$.

When no coefficient is written, 1 is understood.

1. In $8x + 7y + z$, what is the coefficient of $x$? of $y$? $z$?

2. What are the factors of $5m$? $8p$? $29y$?

### DIVISION

**8.** How much is $\frac{1}{4}$ of $12\,a$? Why?

How many times does $12\,a$ contain $3\,a$? Why?

**Division** is the operation of finding a factor when the product and the other factor are given.

The **dividend** is the given product, the **divisor** is the given factor, and the **quotient** is the required factor.

The **sign** of division is $\div$, which is read *divided by*.

Thus, (*a*) $12\,x \div 4\,x = 3$, because $3 \times 4\,x = 12\,x$.

(*b*) $12\,x \div 4 = 3\,x$, because $3\,x \times 4 = 12\,x$.

(*a*) may be read, $12\,x$ contains $4\,x$, 3 times; and (*b*) may be read, $\frac{1}{4}$ of $12\,x$ is $3\,x$.

Hence, by division we find how many times one number is contained in another, and also one of the equal parts of a number.

Division is also indicated by writing the dividend above the divisor with a horizontal line between them.

Thus, $\frac{12}{3} = 12 \div 3 = 4$.

NOTE. At present, a quotient in the form of $\frac{a}{b}$ is to be considered an integer. That is, $a$ is supposed to be divisible by $b$.

Find the values of:

| | | | | |
|---|---|---|---|---|
| **1.** | $15 \times 4$ | $24\,a \div 6$ | $\frac{1}{4}$ of $28\,n$ | $16\,x \div 8\,x$. |
| **2.** | $18 \div 6$ | $25\,b \times 4$ | $\frac{1}{6}$ of $54\,y$ | $39\,y \div 3\,y$. |
| **3.** | $3\,x \times 7$ | $36\,c \div 9$ | $\frac{1}{9}$ of $45\,c$ | $35\,n \div 7\,n$. |
| **4.** | $8\,x \div 4$ | $40\,y \times 6$ | $\frac{1}{7}$ of $42\,a$ | $72\,b \div 8\,b$. |

**5.** $\frac{18}{3}$ $\frac{25\,x}{5\,x}$ $\frac{48\,a}{8\,a}$ $\frac{42\,c}{6\,c}$ $\frac{100\,b}{25}$ $\frac{91\,x}{7}$ $\frac{143\,y}{11\,y}$.

### EXERCISES IN MULTIPLICATION AND DIVISION

**9. 1.** If $a$ stands for 5, what number is represented by $5\,a$? $7\,a$? $20 \div a$?

**2.** If $b = 6$, what is the numerical value of $b \times 8$? $b + 3$? $\frac{b}{2}$?

**3.** How many times greater is $8\,a$ than $a$?

**4.** How many times greater is $7\,c$ than 7?

**5.** One number is $x$ and another is 6 times as great. Find the greater number.

**6.** If a pen costs $a$ cents, how much will 5 pens cost?

**7.** If a pear costs 3¢, how much will $n$ pears cost?

**8.** If a horse walks 6 mi. per hour, how far will it walk in $x$ hours?

**9.** If cheese is worth $5\,x$ cents a pound, how much will 9 lb. cost?

**10.** What will be the cost of 8 pigs at $3\,x$ dollars each?

**11.** What will be the cost of $4\,y$ pigs at $6 each?

**12.** If 8 yd. of cloth cost $24\,y$ cents, how much will 1 yd. cost?

**13.** A lady bought $n$ pounds of butter for $24\,n$ cents. What was the price per pound?

**14.** A boy sold $5\,x$ birds and $3\,x$ squirrels, receiving 9¢ each for the birds and 12¢ each for the squirrels. How much did he receive for all?

**15.** One number is $x$ and another is 3 more than twice as much. Find the greater number.

**16.** A buggy cost $3\,a$ dollars and a horse $10 less than 4 times as much. Find the cost of the horse.

### LAWS OF SIGNS OF OPERATIONS

**10.** 1. How many ones are $7 \times 4$ or 7 fours?

2. How many ones are $\frac{12}{6}$ or 12 sixths?

3. Change $6 \times 3$ and $20 \div 4$ to ones.

4. Find $7 + 4$ or the sum of 7 ones and 4 ones.

5. Find $7 - 4$ or the difference between 7 ones and 4 ones.

6. Find $6 \times 3 + 5$ or the sum of 6 threes and 5 ones.

7. Find $10 \div 2 - 3$, or the difference between 10 halves and 3 ones.

When numbers are connected by the signs $+$, $-$, $\times$, $\div$, *the operations indicated by* $\times$ *and* $\div$ *must be performed before the operations indicated by* $+$ *and* $-$.

Reduce to single numbers or find the values of:

8. $7 \times 4 + 3$ $\quad 12 \times 2 - 6$ $\quad 13 + 3 \times 5$ $\quad 11 - 2 \times 4$.

9. $6 \div 2 + 5$ $\quad 35 \div 5 - 4$ $\quad 7 + 18 \div 6$ $\quad 20 - 9 \div 3$.

Find the values of the following, supposing $x$ to stand for 2 and $y$ for 3:

10. $7x + 5$ $\quad 14y - 6$ $\quad 12x + 3y$ $\quad 9x - 2y$.

11. $\frac{12}{y} - 2$ $\quad 25 - \frac{24}{y}$ $\quad \frac{120}{x} + \frac{15}{y}$ $\quad \frac{24}{x} - \frac{21}{y}$.

**11.** In expressions like these, when we make $x = 2$ and $y = 3$, we are said to **substitute** 2 for $x$ and 3 for $y$.

Find the value of:

12. $6a + 3b$, when 4 is substituted for $a$ and 3 for $b$.

13. $5n - \frac{12}{c}$, when 3 is substituted for $n$ and 4 for $c$.

### NUMERICAL VALUES

**12.** **1.** May $x$ stand for, or represent, any integer?

**2.** If $x$ stands for, or represents, 2, what number does $3x$ represent? $7x$? $5x+8$? $9x-11$? $12 \div x+7$?

When 2 is substituted for $x$, the value of $12 \div x + 7$, namely 13, is called the **numerical value** of $12 \div x + 7$ for $x = 2$.

**3.** If $x = 3$, what is the numerical value of $5x$? $10x - 7$? $30 \div x - 5$? $\frac{36}{x} - \frac{15}{x}$?

**4.** If $x = 12$, what is the value of $\frac{2x}{3}$? $\frac{3x}{4}$? $\frac{5x}{6}$?

Find, in two ways, the values of the following when 6 is substituted for $x$:

**5.** $x + 3x$. **6.** $3x + 7x$. **7.** $2x + 8x + 12x$.

**8.** $2x - \frac{x}{2}$. **9.** $\frac{2x}{3} - \frac{x}{2}$. **10.** $\frac{x}{2} + \frac{2x}{3} + \frac{3x}{2}$.

**11.** Find the value of $5x - 12x + 20x$ when $x = 3$.

**12.** Find the value of $\frac{3x}{4} + \frac{5x}{3} + \frac{7x}{4}$ when $x = 12$.

**13.** Find the sum of $5x$ and $8x$ when $x = 5$.

**14.** Find the difference of $9x$ and $3x$ when $x = 8$.

**15.** Find the product of $5x$ by 9 when $x = 8$.

**16.** Find the quotient of $7x$ by 3 when $x = 9$.

**17.** Show that $3x + \frac{15x}{6} = 165$ when $x = 30$.

**18.** Verify $2x + 8x = 5x + 10$, for $x = 2$.

**19.** Verify $9x - 11 = 6x + 16$, for $x = 9$.

**20.** Verify $7x - 18 = 32 - 3x$, for $x = 5$.

## INTRODUCTION OF EQUATIONS

**13.** 1. If $8x$ is 40, what is the value of $x$? Why?

2. If $9x = 63$, what is the value of $x$?

*Ans.* 7. Because $9x = 63$ is verified for $x = 7$.

3. If $8x = 72$, what is the value of $x$? Why?

4. If $6x = 24$, what is the value of $x$? Why?

5. If $x + 3x = 16$, what is the value of $x$?

FIRST SOLUTION. Substitute for $x$ the numbers 1, 2, 3, etc., until a number is found that satisfies the equation.

SECOND SOLUTION. $x + 3x = 4x$; then $4x = 16$. Therefore, $x = \frac{1}{4}$ of 16, which is 4.

NOTE. The student should use the first method of solution until he thoroughly understands what he is required to find, viz.: *the number which, when substituted for $x$, will give the two members of the equation the same numerical value.*

6. Given $x + 5x = 36$. Find the value of $x$.

7. Given $2x + 5x = 84$. Find the value of $x$.

8. Given $9x - 4x = 35$. Find the value of $x$.

Find the value of $x$ in the following:

9. $x + 2x + 5x = 48$.

10. $12x - 5x = 21$.

11. $x + 2x + 6x = 72$.

12. $14x + 6x = 80$.

13. $x + 3x + 8x = 84$.

14. $20x - 3x = 51$.

Let $x$ be found in the following by trial substitutions:

15. $3x + 7 = 2x + 9$.

16. $7x - 3 = 4x + 9$.

17. $5x - 2 = 3x + 4$.

18. $11 - x = 2x - 1$.

19. $\frac{2x}{3} + 5 = \frac{5x}{6} + 4$.

20. $\frac{3}{4}x + 9 = 11 + \frac{x}{2}$.

Can you find a shorter method of solving Ex. 15–20?

### PROBLEMS

1. One number is 3 times another number. If $x$ stands for the less number, what will stand for the greater? What will stand for their sum? If their sum is 36, what equation may we write? Find from this equation the value of $x$, then of $3x$. What, then, are the numbers? Prove it.

2. One number is 5 times another number, and the sum of the numbers is 42. Find the numbers.

Let $x$ = the less number, then $5x$ = the greater, etc.

3. One number is 8 times another number, and the sum of the numbers is 45. Find the numbers.

4. The sum of two numbers is 35, and one of the numbers is 4 times the other. Find the numbers.

5. The sum of two numbers is 72, and one of the numbers is 11 times the other. Find the numbers.

6. Mary has 2 times as many pins as Lucy, and they together have 48. How many has each?

Let $x =$, or stand for, the number of pins Lucy has;
then $2x =$, or stands for, the number of pins Mary has,
and $x + 2x =$, or stands for, the number of pins they together have.
But 48 is the number of pins they together have.

$$\therefore x + 2x = 48;$$

or $$3x = 48.$$

Then $$x = 16,$$

and $$2x = 32.$$

Therefore, Lucy has 16 pins and Mary 32 pins.

NOTE. The symbol $\therefore$ stands for **therefore**, and is so read.

7. Henry has 3 times as many apples as Frank, and they together have 32. How many has each?

**8.** A father is 5 times as old as his son, and the sum of their ages is 48 yr. Find their ages.

**9.** A horse and saddle are together worth $165, and the horse is worth 10 times as much as the saddle. Find the value of each.

**10.** A lady bought an equal amount of flour and rice. She paid 4¢ a pound for the flour, 7¢ a pound for the rice, and the amount of her bill was $1.65. How many pounds of each did she buy?

**11.** A farmer sold twice as many pigs as calves. He received $5 each for the pigs, $8 each for the calves, and $126 for all. How many of each did he sell?

**12.** A man has $76 in two-dollar bills and five-dollar bills, and he has 7 times as many two-dollar bills as five-dollar bills. How many has he of each?

**13.** Two men, who are 495 mi. apart, are traveling toward each other. If one goes 30 mi. a day, and the other 25 mi. a day, in how many days will they meet?

**14.** A grocer sold 8 lb. of coffee and 3 lb. of sugar for $1.40, asking 4 times as much for a pound of coffee as for a pound of sugar. Find the price per pound of each.

**15.** A farmer sold a number of sheep for $300, $\frac{1}{2}$ of them at $4 each, $\frac{1}{3}$ of them at $5 each, and $\frac{1}{6}$ of them at $8 each. How many sheep did he sell?

HINT. Let $6x$ = the number of sheep he sold.

**16.** B is twice as old as A, C is three times as old as B, and the sum of their ages is 72 yr. Find the age of each.

**17.** Divide $48 among A, B, and C so that B shall have 3 times as much as A, and C as much as A and B together.

**18.** John has \$7 more than Henry, and both together have \$25. How much has each?

Let $x$ = the number of dollars Henry has;
then $x + 7$ = the number of dollars John has,
and $x + x + 7$ = the number of dollars both have.
But $25$ = the number of dollars both have.
Therefore, $x + x + 7 = 25$,
or $2x + 7 = 25$.
Whence, $x = 9$, the number of dollars Henry has,
and $x + 7 = 16$, the number of dollars John has.

**19.** One number is 12 more than another number, and the sum of the numbers is 32. Find the numbers.

**20.** The difference between two numbers is 5, and the sum of the numbers is 31. Find the numbers.

**21.** The sum of two consecutive integers is 35. Find the integers.

HINT. Let $x$ and $x + 1$ stand for the integers.

**22.** The sum of three consecutive integers is 33. Find the integers.

**23.** Divide 19 roses between Lucy and Ann so that Ann shall have 3 less than Lucy.

**24.** Frank has 5 more than twice as many marbles as Ben, and both together have 32 marbles. How many has each?

**25.** C is 6 yr. older than B, B is 3 yr. older than A, and the sum of their ages is 78 yr. Find their ages.

**26.** Divide 36 into two parts such that the greater part shall be 8 less than 3 times the less part.

**27.** A boy paid \$1.70 for three books, paying 10¢ more for the second than for the first, and 15¢ more for the third than for the second. Find the cost of each.

## II. POSITIVE AND NEGATIVE SIMPLE INTEGERS

### INDUCTION

**14** 1. A gain of \$5 and a gain of \$3 is equivalent to a gain of how many dollars?

2. A gain of \$5 and a loss of \$3 is equivalent to a gain of how many dollars?

3. A loss of \$5 and a gain of \$3 is equivalent to a loss of how many dollars?

4. A loss of \$5 and a loss of \$3 is equivalent to a loss of how many dollars?

5. Denoting a gain by the sign + and a loss by the sign −, the above questions and their answers may be written, respectively:

$+\$5$ and $+\$3 = ?$ *Ans.* $+\$8$.
$+\$5$ and $-\$3 = ?$ *Ans.* $+\$2$.
$-\$5$ and $+\$3 = ?$ *Ans.* $-\$2$.
$-\$5$ and $-\$3 = ?$ *Ans.* $-\$8$.

6. A boy goes 7 mi. north from the schoolhouse, and then goes south 2 mi. How far, and in what direction, is he then from the schoolhouse?

North 7 mi. plus south 2 mi. = ?

Denoting north by + and south by −, read the following, and give the equivalents both as to distance and direction:

7. + 15 mi. plus + 3 mi. 8. − 12 mi. plus + 8 mi.
9. + 13 mi. plus − 6 mi. 10. − 14 mi. plus − 6 mi.

**15.** A **positive number** is one whose sign is +.

**16.** A **negative number** is one whose sign is −.

When no sign is written before a number, the sign + is understood.

### ADDITION

**17.** How many cents are $6¢ - 4¢ + 9¢ - 7¢$?

FIRST. $6¢ - 4¢ = 2¢$, $2¢ + 9¢ = 11¢$, $11¢ - 7¢ = 4¢$.

SECOND. There are in all 15 positive ¢'s and 11 negative ¢'s, and in whatever order we unite the given numbers of units the 11 negative units will finally cancel 11 of the positive units, and leave 4 positive units, or $+ 4¢$.

Thus, $6¢ - 4¢ = 2¢$; $9¢ - 7¢ = 2¢$; $2¢ + 2¢ = 4¢$.

This process of uniting numbers according to their signs is called **algebraic addition.**

Hereafter, addition will be understood to have this meaning.

**18.** The signs + and − are not only signs of operations, but they are also signs of **opposite qualities or directions.**

Find the sums of the following, and check the results by taking the given numbers in different orders:

**1.** $2x + x + 6x$.

**2.** $10x - 6x + 23x - 15x$.

**3.** $3y + 7y - y$.

**4.** $34y + 8y - 16y - 12y$.

**5.** $8a - 5a + a$.

**6.** $25a - 9a - 28a + 11a$.

**7.** $b - 8b + 3b$.

**8.** $41b - 5b - 13b - 18b$.

**9.** The sum of $a$ and $+ b$ is $a +$ ?

**10.** The sum of $a$ and $- b$ is $a -$ ?

**11.** $x + y$ is the sum of $x$ and ?

**12.** $a - b$ is the sum of $a$ and ?

Find the sum of:

**13.**
$$\begin{array}{r} 5x + 6 \\ 3x - 4 \\ \underline{2x + 7} \end{array}$$

**14.**
$$\begin{array}{r} 8a - 9 \\ 4a + 6 \\ \underline{-7a + 3} \end{array}$$

**15.**
$$\begin{array}{r} 15b + 2 \\ -9b + 4 \\ \underline{-6b - 9} \end{array}$$

**19.** Expressions like $a, -6b, 5x+7$, etc., are called **algebraic numbers**, or simply **numbers**.

In an algebraic number like $7x-3y+8$, $7x$, $-3y$, and $+8$ are the **terms**. An algebraic number having more than one term is a sum, of which the terms are the addends.

**20.** Two algebraic numbers are said to be **like** when their literal factors are the same; as $5a$ and $9a$, $-7x$ and $+3x$.

**21.** The **absolute value** of an algebraic number is its value without reference to its sign.

Thus, the absolute value of $-6$ is 6; of $-4x$, $4x$.

**22.** We may now give the rule for algebraic addition:

*To add two like numbers having the same sign, add their absolute values and give the sum the common sign of the addends.*

*To add two like numbers having different signs, subtract their absolute values and give the difference the sign of the greater addend.*

*To add unlike numbers, write them one after the other, giving each its proper sign.*

Find the sum of:

1. 8, 4, and $-3$.
2. $5x-8y+6$ and $3x+5y-6$.
3. $a$, $b$, and $-c$.
4. $9a+6b+c$ and $3a-9b+c$.
5. $c$, $-d$, and $x$.
6. $7c-9d-8$ and $4d+6-3c$.
7. $6x-8y+9$, $3x+2y-5$, $4x+6y-4$.
8. $9a+7b-c$, $4a+6b-c$, $c-7a-9b$.
9. $5n-6r+8s$, $6n-7s$, $4r-5s+3$.
10. $4c+7d$, $-3d+5e$, $6e-7f$, $6f-4c$.
11. $3a-6b+7c-8d$, $5b-4c+8d-3a$.

### ADDITION PROBLEMS

**23.** **1.** A boy paid $2x$ dollars for a vest, $4x$ dollars for trousers, and $5x$ dollars for a coat. How much did he pay for all? how much if $x = 2$?

**2.** A farmer sold, $8y$ pecks of corn, $11y$ pecks of wheat, and $15y$ pecks of oats. How many pecks did he sell in all? how many if $y = 10$?

**3.** A man went north $75x$ yards, then south $43x$ yards, then south $58x$ yards. How far, and in what direction, was he then from the starting point?

**4.** What is the answer to problem 3 if $x = 25$?

**5.** If $n$ is an even integer, what is the next odd integer? the next even integer?

**6.** What are the answers to problem 5 if $n = 12$?

**7.** A boy rode $m$ miles one day, $n$ miles the next, and 15 miles the next. How far did he ride in all?

**8.** What is the answer to problem 7 if $m = 20$ and $n = 18$?

**9.** Ada earned $a$ cents Monday, $b$ cents Tuesday, and spent 35¢ Wednesday? How much did she then have?

**10.** What is the answer to problem 9 if $a = 75$ and $b = 65$?

**11.** John weighs $x$ pounds, Henry 10 pounds more than John, and James 6 pounds more than Henry. How many pounds do they all weigh together?

**12.** What is the answer to problem 11 if $x = 90$?

**13.** The second of three numbers is 5 more than the first and the third is 8 more than the second. If the first is $x$, what is the sum of the three?

## SUBTRACTION

**24.** How much is 15 less 9? Why?

Which is the minuend? the subtrahend? difference?

The **difference** between one number (called the minuend) and another (called the subtrahend) is the number which added to the subtrahend will give the minuend.

Thus, $11 - 4 = 7$, because $4 + 7 = 11$.

**25. Subtraction** is the process of finding the difference between two numbers, or of taking one number from another.

**26. Law of signs in subtraction.**

By the definition of difference, § 24, we have

(*a*) The difference between $+5$ and $+3$ is $+2$, because $+2$ added to $+3$ is $+5$.

(*b*) The difference between $+5$ and $-3$ is $+8$, because $+8$ added to $-3$ is $+5$.

(*c*) The difference between $-5$ and $+3$ is $-8$, because $-8$ added to $+3$ is $-5$.

(*d*) The difference between $-5$ and $-3$ is $-2$, because $-2$ added to $-3$ is $-5$.

Note that in each case the correct result can be obtained *by changing the sign of the subtrahend, and then adding it to the minuend.*

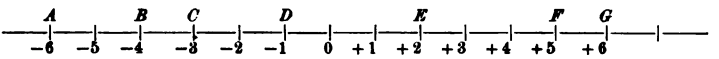
Thus, $+5$ less $+3 = +5$ plus $-3 = +2$;

$+5$ less $-3 = +5$ plus $+3 = +8$;

$-5$ less $+3 = -5$ plus $-3 = -8$;

$-5$ less $-3 = -5$ plus $+3 = -2$.

The principle and process are illustrated in what follows on the next page.




**27.** The signs + and − denote opposite qualities or directions. Hence, if distances laid off from 0 to the right are positive, then distances laid off from 0 to the left are negative.

Now note carefully that the difference between two algebraic numbers, as +5 and +2, is the number of units of distance between them on the scale, 3, and is + or − according as the minuend is on the right or left of the subtrahend. Thus:

1. +6 less +2 = +6 plus −2 = +4, because $G$ is 4 units to the right of $E$.
2. +5 less −3 = +5 plus +3 = +8, because $F$ is 8 units to the right of $C$.
3. −4 less +2 = −4 plus −2 = −6, because $B$ is 6 units to the left of $E$.
4. −6 less −1 = −6 plus +1 = −5, because $A$ is 5 units to the left of $D$.

Subtract the following and check the results by thinking of the minuend, subtrahend, and remainder as distances on the above scale.

| | | | |
|---|---|---|---|
| 1. $+11$ | 2. $+7$ | 3. $-13$ | 4. $-9$ |
| $+5$ | $+10$ | $-8$ | $+6$ |
| 5. $-6x$ | 6. $+5y$ | 7. $-10a$ | 8. $+8b$ |
| $-12x$ | $-8y$ | $+6a$ | $+12b$ |
| 9. $+15n$ | 10. $+16x$ | 11. $-9c$ | 12. $-12d$ |
| $+8n$ | $-7x$ | $-17c$ | $+18d$ |

In the following consider + as meaning above zero, − as meaning below zero, and explain the results by means of a thermometer.

Find the difference:

1. Between $+9°$ and $+3°$; between $+3°$ and $-8°$.
2. Between $-5°$ and $+4°$; between $-6°$ and $-9°$.

In these and similar problems what is meant by a positive answer? a negative answer?

3. The difference between 8 and $+5$ is $8-$?
4. The difference between $a$ and $+b$ is $a-$?
5. The difference between 9 and $-6$ is $9+$?
6. The difference between $a$ and $-b$ is $a+$?
7. $x-y$ is the difference between $x$ and ?
8. $x+y$ is the difference between $x$ and ?
9. What is the difference between 8 and $+a$?
10. What is the difference between 6 and $-a$?

Find the difference:

11. Between $8x$ and $11x$; between $+3y$ and $-6y$.
12. Between $17x$ and $9x$; between $-5a$ and $+8a$.
13. Between $6a+7b-5$ and $3a-2b-9$.

$$\begin{array}{r} 6a+7b-5 \\ -3a+2b+9 \\ \hline 3a+9b+4, \quad Ans. \end{array} \qquad \begin{array}{r} 6a+7b-5 \\ 3a-2b-9 \\ \hline 3a+9b+4, \quad Ans. \end{array}$$

In both solutions the terms of the subtrahend are written under like terms of the minuend.

In the first solution we changed the signs of the subtrahend and proceeded as in addition. In the second solution we simply conceived the signs of the subtrahend to be changed, and proceeded as in addition. The latter method is preferable.

**14.** From $8a - 2b - 7c$ take $3a - 2b - 4c$.

**15.** From $9x + 7y - 9z$ take $5x - 2y - 9z$.

**16.** From $3a + 6c + 5d$ take $6a - 3c + 8d$.

**17.** From $7n - 5p - 18$ take $3n - 9p + 15$.

**18.** From the sum of $5x + 4$ and $3x - 2$ take their difference.

**19.** How much more is $6a + 3b$ than $4a - 2b$?

**20.** How much more is $7x + 15$ than $7x - 15$?

### PARENTHESES

**28.** Find the value of $(6x + 2y) + (3x - 7y)$.

This means the sum of $6x + 2y$ and $3x - 7$.
Hence, it may be written $6x + 2y + 3x - 7y$.

*Parentheses preceded by +, or by no sign, can be removed without changing the sign of any term inside.*

Find the value of $(6x + 2y) - (3x - 7y)$.

This means the difference between $6x + 2y$ and $3x - 7y$.
Hence, it may be written $6x + 2y - 3x + 7y$.

*Parentheses preceded by the sign — can be removed by changing the sign of every term inside.*

Find the value of:

**1.** $(x+y)+(x-y)$. **2.** $(3a+2b+5)+(a-3b+1)$.

**3.** $(x+y)-(x-y)$. **4.** $(5a-4b-7)-(4a+b-6)$.

**5.** $(6+x)+(5-x)$. **6.** $(8c+5d-3)-(3c-d-3)$.

**7.** $(5-3)-(3-5)$. **8.** $(6x-7y-2)-(8x+y+5)$.

**9.** $+(5x-7x+4)$. **10.** $-(6a+3b-8a-5b+3)$.

**11.** $-(14-12-3)$. **12.** $-(10-17+15-21-6)$.

**13.** $-(+4)$; $-(-6)$; $+(+3)$; $+(-1)$; $-(-1)$.

### SUBTRACTION PROBLEMS

**29.** **1.** A boy sold a pony for \$70 and gained \$3. How much did the pony cost? How much would it have cost if the gain had been $a$ dollars?

**2.** The sum of two numbers is 25. What is the second number if the first is 16? if the first is $m$?

**3.** The greatest of three consecutive integers is 15. What are the other two? What are the other two if the greatest is $n$?

**4.** Mary bought a hat for \$5, a bag for \$2, and gave in payment a 10-dollar bill. How much change was due her? How much if the cost of the hat was $a$ dollars?

**5.** A merchant bought a gun for \$18 and a shot pouch for \$4, and sold the two for \$35. How much did he gain? How much would he have gained if the gun had cost $m$ dollars and the shot pouch $n$ dollars?

**6.** Ann had $c$ cents and gave 25¢ for a doll. How many cents had she left?

**7.** Ned paid $b$ dollars for a book and $g$ dollars for a gun. How much more did the gun cost than the book?

**8.** A lady sold a cow for \$50 and gained $a$ dollars. How much would she have gained if she had sold the cow for $m$ dollars?

**9.** James and John each bought a peck of berries, James paying 40¢ and John $n$ cents. James sold his for $m$ cents and John his for 75¢. How much more did James gain than John?

**10.** A man paid \$25 for $c$ cords of wood and sold it for \$6 a cord. How much did he gain?

## MULTIPLICATION

**30.** How much is $6 \times 5x$?

Which is the multiplicand? the multiplier? the product?

What does the multiplier show?

*The muliplier shows also, by its sign, how the multiplicand is to be taken, whether additively or subtractively.*

**31. Law of signs in multiplication.**

1. Find $+3 \times +2$.

$+3 \times +2 = +(3 \times +2) = +(+2+2+2) = +(+6) = +6.$

2. Find $+3 \times -2$.

$+3 \times -2 = +(3 \times -2) = +(-2-2-2) = +(-6) = -6.$

3. Find $-3 \times +2$.

$-3 \times +2 = -(3 \times +2) = -(+2+2+2) = -(+6) = -6.$

4. Find $-3 \times -2$.

$-3 \times -2 = -(3 \times -2) = -(-2-2-2) = -(-6) = +6.$

That is,

$$+3 \times +2 = +6, \quad +3 \times -2 = -6,$$
$$-3 \times +2 = -6, \quad -3 \times -2 = +6.$$

From which we infer:

*Like signs in multiplication give* $+$.

*Unlike signs in multiplication give* $-$.

Or, for short, we may say:

$+$ *multiplied by* $+$ *is* $+$; $+$ *multiplied by* $-$ *is* $-$; $-$ *multiplied by* $-$ *is* $+$; $-$ *multiplied by* $+$ *is* $-$.

1. Multiply $+6$ by $+3$; $-8$ by $+4$; $+9$ by $-x$.
2. Multiply $-5$ by $-4$; $+7$ by $-3$; $-y$ by $+5$.
3. Multiply $-6$ by $+a$; $-b$ by $-6$; $+4$ by $-c$.

4. Multiply $+8$ by $+5$; $+7x$ by $+3$.
5. Multiply $+9$ by $-4$; $+6y$ by $-5$.
6. Multiply $-7$ by $+6$; $-8a$ by $+6$.
7. Multiply $-8$ by $-9$; $-5b$ by $-7$.

**32.** If the sign of the multiplicand indicates a certain *direction* or *quality*, the product will have the same or the opposite direction or quality according as the sign of the multiplier is $+$ or $-$.

Explain the following on the supposition that $+$ before the multiplicand means *north* and $-$ *south*:

1. Multiply $+5$ miles by $+3$; $+6x$ miles by $+2$.
2. Multiply $+8$ miles by $-4$; $+7y$ miles by $-5$.
3. Multiply $-6$ rods by $+9$; $-5n$ rods by $+4$.
4. Multiply $-9$ rods by $-4$; $-8a$ rods by $-7$.

Explain the following on the supposition that $+$ before the multiplicand means *gain* and $-$ *loss*.

Find the value of:

5. $+12¢ \times +5$.
6. $+\$15 \times -4$.
7. $-25¢ \times +3$.
8. $-\$50 \times -6$.
9. $+6¢ \times -5$.
10. $-8n$ cents $\times -4$.
11. $-3c$ cents $\times -7$.
12. $+9x$ cents $\times +7$.

Explain the following on the supposition that $+$ before the multiplicand means *forward* and $-$ *backward*:

13. $+9$ ft. $\times -5$.
14. $+6a$ rods $\times +7$.
15. $-8$ yd. $\times +6$.
16. $-7n$ rods $\times -4$.

A **positive** multiplier **preserves** the quality of the multiplicand, and a **negative** multiplier **reverses** it.

### THE DISTRIBUTIVE LAW OF MULTIPLICATION

**33.** Find the product of $a + b - c$ by 3.

$$\begin{array}{l} 1 \text{ time } a+b-c = \quad a+\quad b-\quad c \\ 1 \text{ time } a+b-c = \quad a+\quad b-\quad c \\ \underline{1 \text{ time } a+b-c = \quad a+\quad b-\quad c} \\ 3 \text{ times } a+b-c = 3a+3b-3c \end{array}$$

This is usually written:

$$3 \times (a+b-c) \text{ or } 3(a+b-c) = 3a+3b-3c.$$

*The product of an algebraic sum by a number is equal to the algebraic sum of the products of the addends by the number.*

Find the values of:

**1.** $6 \times (2a+3b-2c)$. **2.** $5(3x-2y+10)$.

**3.** $4 \times (5a-2x-6y)$. **4.** $7(8c+5d-3x)$.

**5.** $(3x-4y-6) \times -3$. **6.** $(9a-3n+7) \times -8$.

**7.** $(5a+3b-7) \times -4$. **8.** $(6e-5f-8) \times -5$.

**9.** $3(5x-2x+4x)$.

This may be solved in two ways, viz.:

(1) $5x-2x+4x = 7x$; $3 \times 7x = 21x$.

(2) $3(5x-2x+4x) = 15x-6x+12x = 21x$.

Find the following in two ways, and see that the results are the same:

**10.** $4(9+6-5)$. **11.** $(8x-3x-2x) \times -5$.

**12.** $5(8-3-2)$. **13.** $(7a+4a+3a) \times -6$.

**14.** $a(6+8-4)$. **15.** $(9b-8b-6b) \times +7$.

**16.** $b(7-6-9)$. **17.** $(6c-9c+3c) \times -4$.

**18.** $5(10x-4+7x-8-12x+5)$.

**19.** $6(15a+7-18a+6+7a-3)$.

**20.** $7(9a-3b+5-3a-4b-1)$.

### MULTIPLICATION PROBLEMS

**34.** **1.** One number is 3 times another. If the less is $x$, what is the sum of the two numbers?

**2.** The first of three numbers is $x$; the second is 4 times the first, and the third is 7 times the second. Find the sum of the three numbers.

**3.** A farmer sold $m$ hogs at \$7 each and $n$ sheep at \$8 each. How much did he receive for all?

**4.** A dealer bought 5 horses at $m$ dollars each and 7 mules at $n$ dollars each. How much more did the horses cost than the mules?

**5.** The less of two numbers is $x$ and the greater is 5 more than 3 times the less. Find the sum of the numbers.

**6.** A farmer has $m$ sheep and 3 times as many hogs less 12. How many has he in all?

**7.** Find the cost of $8m$ clocks at \$6 each.

**8.** A man earns \$25 per week. If his expenses are $n$ dollars per week, how much money will he save in 4 weeks?

**9.** A farmer sold 40 pigs. For $n$ of them he received \$3 each, and for the remainder \$2 each. How much did he receive for all?

**10.** From a bolt of cloth containing 40 yards there were sold $m$ yards, and then 5 yards more. What was the value of the remainder at 8¢ a yard?

**11.** How far will a horse walk in 7 hours at the rate of 6 mi. per hour? in $n$ hours at 5 mi. per hour?

**12.** A boy starts on his bicycle to go 100 mi. After going $x$ hours at 8 mi. per hour, then $y$ hours at 7 mi. per hour, how many miles will he still have to go?

### DIVISION

**35.** How much is 20 divided by 4? Why?

Which is the dividend? the divisor? the quotient?

The **quotient** is the number which multiplied by the divisor will give the dividend.

Thus, $12 \div 3 = 4$, because $3 \times 4 = 12$.

Again, $20\,c \div 4\,c = 5$, because $5 \times 4\,c = 20\,c$.

When we take one of the equal parts of a number the operation is called **partition**, and when we find how many times one number is contained in another the operation is called **measuring**.

Division includes both of these operations.

Since $20\,c \div 4\,c = 5$, the $c$'s are said to *cancel* each other.

1. Divide 18 by 6; 27 by 3; $6x$ by 6; $11x$ by $x$.
2. Divide 35 by 5; 42 by 7; $8x$ by 4; $12x$ by $2x$.
3. Divide 45 by 9; $9a$ by 3; $6b$ by 2; $24c$ by $3c$.
4. Divide $4a$ by 2; $8b$ by 4; $7c$ by $c$; $56y$ by $8y$.

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5. How many times does $36x$ contain $4x$?
6. How many times does $54a$ contain $9a$?
7. How many times does $30n$ contain $6n$?
8. How many times does $72e$ contain $9e$?

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9. How much is $\frac{1}{4}$ of $20x$? $\frac{1}{5}$ of $30a$?
10. How much is $17c \div c$? $25n \div n$?
11. How much is $\frac{1}{8}$ of $40n$? $\frac{1}{7}$ of $63b$?
12. How much is $32n \div 2n$? $84y \div 3y$?

**36. Law of signs in division.**

By the definition of quotient, § 35, we have

$(a)$ $+18 \div +6 = +3$, because $+3 \times +6 = +18$. (1)

$(b)$ $+18 \div -6 = -3$, because $-3 \times -6 = +18$. (2)

$(c)$ $-18 \div +6 = -3$, because $-3 \times +6 = -18$. (3)

$(d)$ $-18 \div -6 = +3$, because $+3 \times -6 = -18$. (4)

From (1) and (4) we infer: *Like signs in division give* $+$.

From (2) and (3) we infer: *Unlike signs in division give* $-$.

*Or, for short, we may say:* $+ \div +$ *is* $+$; $+ \div -$ *is* $-$; $- \div +$ *is* $-$; $- \div -$ *is* $+$.

Note that the law of signs in division is the same as the law of signs in multiplication.

1. Divide $+20x$ by $+4$; $+96x$ by $+6x$.
2. Divide $+28y$ by $-7$; $-42a$ by $-3a$.
3. Divide $+50a$ by $-5$; $+76c$ by $-4c$.
4. Divide $-72c$ by $+8$; $-98y$ by $+7y$.
5. Find $21 \div -3$; $-39 \div -3$; $-156y \div 13$.
6. Find $68 \div +4$; $-8c \div +4$; $425x \div -17$.

Division is also indicated by writing the dividend above the divisor with a line between, the sign of the quotient being written before the dividing line.

Thus, the quotient of $+a$ by $-b$ is $-\frac{a}{b}$.

7. Divide $+a$ by $+n$; $-5b$ by $+7c$; $-3$ by $-5$.
8. Divide $-c$ by $-d$; $+9x$ by $-5y$; $-6$ by $+7$.
9. Divide $+x$ by $-y$; $-3c$ by $+8x$; $+2$ by $-9$.
10. How much is $+13n \div -17c$? $-\frac{1}{3}$ of $+5c$?
11. How much is $-18b \div -23x$? $+\frac{1}{c}$ of $-6x$?

### THE DISTRIBUTIVE LAW IN DIVISION

**37.** By the distributive law of multiplication (§ 33),

$$(3a + 2b - 4c) \times 5 = 15a + 10b - 20c.$$

Hence, $(15a + 10b - 20c) \div 5 = 3a + 2b - 4c$.

*To divide an algebraic number of more than one term by a number, divide each term by the number and take the algebraic sum of the quotients.*

Divide $12x - 9y + 15$ by $-3$. $\quad -3)\overline{12x - 9y + 15}$

$-4x + 3y - 5$ *Ans.*

Find the following:

1. $(16x - 20) \div +4$.
2. $(42a - 14b + 105) \div +7$.
3. $(24a - 30) \div -6$.
4. $(78c - 45d - 87x) \div +3$.
5. $(40c - 35) \div +5$.
6. $(152 - 96x - 272) \div -8$.
7. $(28x - 21) \div -7$.
8. $(54b + 27c - 702) \div +9$.

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9. $\frac{1}{2}$ of $(26y - 14)$.
10. $\frac{1}{5}$ of $(45x - 65y + 85)$.
11. $\frac{1}{7}$ of $(56x + 84)$.
12. $\frac{1}{8}$ of $(64a - 368 - 8c)$.

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Find the following in two ways and see that the results are the same:

13. $\frac{1}{4}$ of $(16c - 12 - 28c + 40)$.

(1) $\frac{1}{4}$ of $(16c - 12 - 28c + 40) = 4c - 3 - 7c + 10 = -3c + 7$.

(2) $\frac{1}{4}$ of $(16c - 12 - 28c + 40) = \frac{1}{4}$ of $(-12c + 28) = -3c + 7$.

14. $\frac{1}{3}$ of $(12 - 18 + 27)$.
15. $(25x - 60x - 175x) \div +5x$.
16. $\frac{1}{6}$ of $(30 - 42 - 24)$.
17. $(78c + 42c - 384c) \div -6c$.
18. $\frac{1}{3}$ of $(9c + 6c - 3c)$.
19. $(96a - 504 - 156a) \div -12$.
20. $(16x - 24 - 12y + 36 - 40x - 32y + 64) \div +4$.
21. $(42a + 28b - 84 + 56b + 63 - 70a - 7b) \div -7$.

### DIVISION PROBLEMS

1. If 4 pears cost $n$ cents, how much does 1 pear cost?

2. What is the quotient of $a$ divided by $n$?

3. How many pens at 2 cents each can be bought for $m$ cents?

4. Charles has $a$ marbles, Frank has $b$ marbles, and Thomas $\frac{1}{2}$ as many as Charles and Frank together. How many has Thomas?

5. Three boys weigh $a$ pounds, $b$ pounds, and $c$ pounds respectively. Find the average of their weights.

6. Robert swam $m$ feet and Gordon swam $\frac{1}{4}$ farther than Robert. How many feet did both swim together?

7. How many days will it take a man to earn 40 dollars, if he receives $a$ dollars per day?

8. The sum of two numbers is $s$, and one of the numbers is $x$. Find $\frac{1}{5}$ of the other number.

9. In a class of 10 boys, the average age of 4 of the boys is $m$ years and of the remainder $n$ years. Find the average age of the class.

10. A grocer mixed two kinds of coffee, taking 12 lb. worth $a$ cents per pound and 8 lb. worth $b$ cents per pound. Find the value of one pound of the mixture.

11. A boy is reading a book of 360 pages, and has read $n$ pages. How many pages must he read per day to finish the book in 8 more days?

12. A lady bought a piano for \$500, agreeing to pay for it in monthly payments of $m$ dollars each. If she has made 10 payments, how many more payments will she have to make?

### EQUATIONS OF ONE UNKNOWN NUMBER

**38.** The numerical value of $13 - 4$ is 9. The numerical value of $6 + 3$ is also 9. Therefore we may write

$$13 - 4 = 6 + 3,$$

which is called an *equation.*

An **equation** is a statement that the two members have the same numerical value.

That which precedes the sign of equality is called the **first member** of the equation; and that which follows the sign of equality, the **second member.**

Compare the two members of the following equations:

1. $16 - 5 = 7 + 4$.
2. $33 + 10 = 24 + 19$.
3. $11 + 6 = 9 + 8$.
4. $33 + 24 = 79 - 22$.

What must be the value of $x$ in order that the requirement may be met in the following:

5. $8 + x = 13$. Since $8 + 5 = 13$, $x$ must equal 5.
6. $6 + x = 14$.
7. $13 - x = 9$.
8. $7 + x = 15$.

In an equation like $9 + x = 16$, there are two numbers whose value are **known**, 9 and 16, and there is one number whose value is **unknown**, $x$.

**39. Unknown numbers** are letters used to represent numbers whose values are to be found.

To **solve** an equation is to find the value of the unknown number.

Let the student solve the following by substituting 1, 2, 3, etc., for $x$ until the right value is found.

9. $3x + 24 = 79 - 2x$.
10. $24x - 49 = 19x - 14$.
11. $7x + 1 = 8x + 19$.
12. $11x + 10 = 8x + 19$.

**40.** The direct solution of equations depends on a few principles which will now be illustrated.

Take the equation $16 - 4 = 10 + 2$.

$$(a)\quad \begin{array}{r} 16-4=10+2 \\ +2 \qquad +2 \\ \hline 16-2=10+4 \end{array} \qquad (b)\quad \begin{array}{r} 16-4=10+2 \\ -2 \qquad -2 \\ \hline 16-6=10 \end{array}$$

$$(c)\quad \begin{array}{r} 16-4=10+2 \\ \times 2 \qquad \times 2 \\ \hline 32-8=20+4 \end{array} \qquad (d)\quad \begin{array}{r} 16-4=10+2 \\ +2 \qquad \div 2 \\ \hline 8-2=5+1 \end{array}$$

In $(a)$ 2 is added to both members. Note that the sums are equal.

In $(b)$ 2 is subtracted from both members. Note that the remainders are equal.

In $(c)$ both members are multiplied by 2. Note that the products are equal.

In $(d)$ both members are divided by 2. Note that the quotients are equal.

### AXIOMS

**41.** I. *If the same number or equal numbers are added to, or subtracted from, both members of an equation, the results will be equal.*

**42.** II. *If both members of an equation are multiplied or divided by the same number or by equal numbers, the results will be equal.*

It follows from II that the signs of all the terms of an equation may be changed without destroying the equality, since changing the signs of all the terms of the equation is equivalent to multiplying both members by $-1$.

**43.** 1. Solve the equation $5x = -20$.

Divide both members by 5, § 42. The result is $x = -4$, *Ans.*

2. Solve the equation $9x - 2x = 13 + 8$.

Unite like terms. The result is $7x = 21$.

Divide both members by 7. The result is $x = 3$, *Ans.*

To **verify** or **check** this solution we must substitute 3 for $x$ in the first equation and see that the two members have the same numerical value.

Thus, substituting 3 for $x$ in $9x - 2x = 13 + 8$, we have $9 \times 3 - 2 \times 3 = 27 - 6 = 21$, and $13 + 8 = 21$. Therefore the solution is correct.

Solve, and check, the following:

1. $6x = 42$.
2. $6x + 2x = 14 - 6$.
3. $5x = 45$.
4. $8x - 3x = 42 + 8$.
5. $8x = 64n$.
6. $9x + 5x = 32 - 4$.
7. $4x = 88m$.
8. $7x + 3x = 52 + 8$.
9. $7x = -35a$.
10. $5x - 4x = 13 + 4$.
11. $9x = -72b$.
12. $7x + 5x = 85 - 1$.
13. $6x = 13 + 11$.
14. $5x + 4x = 33a + 12a$.
15. $5x = 14 + 16$.
16. $8x - 5x = 19a + 14a$.
17. $8x = 52 - 12$.
18. $4x + 3x = 55b - 13b$.
19. $3x = 32 - 41$.
20. $6x + 9x - 2x = 41 - 15$.
21. $6x = 15 - 57$.
22. $7x - 8x + 6x = 17 - 42$.
23. $5x + 3x = 24$.
24. $6x + 5x + 4x = 29 + 31$.
25. $6x - 2x = 32$.
26. $9x - 2x - 3x = 63 - 19$.
27. $9x - 4x = 50$.
28. $4x - 7x - 5x = 43 + 13$.
29. $3x - 9x = 36$.
30. $5x - 8x = 25 - 41 - 11$.
31. $x - 12x = 77$.
32. $9x - 3x = 17 + 52 - 27$.

## TRANSPOSITION

**44. Transposing** is transferring a term from one member of an equation to the other.

Find the value of $x$ in $9x - 7 = 4x + 3$.

(1) $9x - 7 = 4x + 3$

$\quad +7 \qquad +7$

(2) $9x \quad = 4x + 3 + 7$

$\quad 4x \qquad 4x$

(3) $9x - 4x = 3 + 7$

(4) $\quad 5x = 10,$

(5) $\quad x = 2.$

In solving an equation we desire to get all the $x$'s in the first member and all the known terms in the second member.

By adding $+7$ to both members of equation (1) we get equation (2), § 41; and by subtracting $4x$ from both members of equation (2) we get equation (3), § 41.

By these processes 7 is transposed from the first to the second member, and $4x$ from the second to the first member, each having its original sign changed.

**45.** *Any term may be transposed from one member to the other, provided its sign is changed from + to − or from − to +.*

Solve, and check, the following:

1. $2x + 11 = 15.$
2. $9x + 5 = 5x + 23 + 2x.$
3. $3x - 10 = 14.$
4. $4x - 6x - 8 = 8 + 3 - 3x.$
5. $5x - 12 = 18.$
6. $9x + 15 = 26 + 2x + 3.$
7. $4x + 13 = 37.$
8. $4x + 5 - 3x = 17 - 2x.$
9. $7x - 15 = 20.$
10. $8x - 3 + 2x = 6x + 21.$
11. $4x = 35 - 3x.$
12. $3x - 19 = 21 - 10x + 12.$
13. $5x = 60 + 2x.$
14. $6x - 5 - 30 = 10 - 5x - 4x.$
15. $8x = 4n + 6x.$
16. $8x + 30 + 10 + 16 = -6x.$
17. $7x = 4x + 9a.$
18. $25 - 5x + 26 = 5x - 49.$
19. $9x = x + 24b.$
20. $17 - 2x + 3x = 14x - 22.$

### EQUATIONS OF TWO UNKNOWN NUMBERS

**46.** In the equation $5x + 2y = 16$ substitute 2 for $x$ and 3 for $y$. Does it give to both members the same numerical value?

If it does, then $x = 2$ and $y = 3$ are said to **satisfy** the equation.

Show that the equation $3x - 2y = 2$ is satisfied for $x = 2, y = 2$; $x = 4, y = 5$; $x = 6, y = 8$; $x = 8, y = 11$.

In the two equations $2x - y = 3$, (1)

and $x + 3y = 19$, (2)

(*a*) Show that $x = 2$ and $y = 1$ will satisfy (1) but not (2).

(*b*) Show that $x = 10$ and $y = 3$ will satisfy (2) but not (1).

(*c*) Show that $x = 4$ and $y = 5$ will satisfy both (1) and (2).

The values of $x$ and $y$ that satisfy both equations may be found thus:

| | | |
|---|---|---|
| Multiply (1) by 3, | $6x - 3y = 9.$ | § 42 |
| Write (2) beneath, | $x + 3y = 19.$ | |
| Add, | $7x = 28.$ | § 41 |
| | $\therefore x = 4.$ | § 42 |
| Again, multiply (2) by 2, | $2x + 6y = 38.$ | § 42 |
| Write (1) beneath, | $2x - y = 3.$ | |
| Subtract, | $7y = 35.$ | § 41 |
| | $\therefore y = 5.$ | § 42 |

**47. Simultaneous equations,** like (1) and (2), are such as can be satisfied for the same values of the unknown numbers $x$ and $y$.

**48.** To **solve** simultaneous equations is to find the values of the unknown numbers that will satisfy the equations.

Solve the equations $\begin{cases} 5x + 4y = 7, & (1) \\ 2x - 3y = 12. & (2) \end{cases}$

In what follows (1), (2), (3), etc., mean and are called, respectively, equation one, equation two, equation three, etc.

To find the value of $x$ we must combine (1) and (2) so as to eliminate, or get rid of, $y$. The combinations are made by the principles stated on page 37.

To make the coefficient of $y$ numerically the same in both equations we may multiply (1) by the coefficient of $y$ in (2), and (2) by the coefficient of $y$ in (1). Thus:

Multiplying (1) by 3 gives $15x + 12y = 21$. § 42 (3)

Multiplying (2) by 4 gives $8x - 12y = 48$. (4)

Adding (3) and (4) gives $23x = 69$. § 41

$\therefore x = 3$. § 42

**49.** This process is called **elimination by addition**, because we get rid of $y$ by *adding* the equations.

Again, to find the value of $y$ we may eliminate $x$ from (1) and (2). To do so we may multiply (1) by the coefficient of $x$ in (2), and (2) by the coefficient of $x$ in (1). Thus:

Multiplying (1) by 2 gives $10x + 8y = 14$. § 42 (5)

Multiplying (2) by 5 gives $10x - 15y = 60$. (6)

Subtracting (6) from (5) gives $23y = -46$. § 41

$\therefore y = -2$.

**50.** This process is called **elimination by subtraction**, because we get rid of $x$ by *subtracting* one equation from another.

Having found the value of one of the unknown numbers, it is generally much easier to find the value of the other by substitution.

Thus, substituting 3 for $x$ in (1), we have

$$15 + 4y = 7; \text{ whence, } 4y = -8, \text{ and } y = -2.$$

Find the values of $x$ and $y$ in the following:

1. $\begin{cases} x+5y=11, \\ 5x-2y=1. \end{cases}$

2. $\begin{cases} 2x-3y=7, \\ 5x+2y=27. \end{cases}$

3. $\begin{cases} 3x-2y=7, \\ 5y-2x=10. \end{cases}$

4. $\begin{cases} 3x-2y=18, \\ 4x+3y=75. \end{cases}$

5. $\begin{cases} 3x+2y=19, \\ 2x-3y=4. \end{cases}$

6. $\begin{cases} 5x+4y=160, \\ 3x+5y=135. \end{cases}$

7. $\begin{cases} 5x-4y=8, \\ 7x+3y=80. \end{cases}$

8. $\begin{cases} 5x+9y=60, \\ 7x+4y=41. \end{cases}$

9. $\begin{cases} 3x+8y=31, \\ 9x+2y=49. \end{cases}$

10. $\begin{cases} 6x-13y=-1, \\ 5x-12y=-2. \end{cases}$

11. $\begin{cases} x+3y=11, \\ 3x+y=17. \end{cases}$

12. $\begin{cases} x+4y=22, \\ 4x-y=37. \end{cases}$

13. $\begin{cases} 4x-3y=24, \\ 3x-2y=19. \end{cases}$

14. $\begin{cases} 4x+7y=29a, \\ 5x+9y=37a. \end{cases}$

15. $\begin{cases} 5x+3y=46, \\ 3x+2y=29. \end{cases}$

16. $\begin{cases} 5x-9y=41b, \\ 7x-8y=62b. \end{cases}$

17. $\begin{cases} 2x+y=32n, \\ x+2y=40n. \end{cases}$

18. $\begin{cases} 11x+5y=-43c, \\ 12x-7y=-22c. \end{cases}$

## PROBLEMS INVOLVING EQUATIONS OF TWO UNKNOWN NUMBERS

1. The sum of two numbers is 34, and their difference is 12. Find the numbers.

HINT. Let $x$ and $y$ = the numbers, then $x + y = 34$, $x - y = 12$.

2. The sum of two numbers is 28, and the sum of the greater and 3 times the less is 46. Find the numbers.

3. The difference of two numbers is 8, and the sum of 2 times the greater and 5 times the less is 65. Find the numbers.

4. A boy has 23 oranges and apples. If he sells the oranges for 5¢ each and the apples for 3¢ each, he will receive 97¢ for all. How many of each has he?

5. If 4 pigs and 5 sheep cost $60, and 3 pigs and 2 sheep cost $31 at the same price, what is the price per head of the pigs and the sheep?

6. If 6 lb. of sugar and 7 lb. of coffee cost $1.35, and 5 lb. of sugar and 8 lb. of coffee cost $1.45 at the same rate, what is the price per pound of the sugar and the coffee?

7. In 6 hr. John rides 18 mi. farther than Frank rides in 5 hr., and in 8 hr. Frank rides 24 mi. farther than John rides in 3 hr. How many miles does each ride per hour?

8. If 2 times A's money is added to 3 times B's money, the sum is $93, but if 5 times A's is added to 4 times B's, the sum is $159. How much money has each?

9. A horse and buggy together cost $243, and the horse cost $18 more than twice as much as the buggy. Find the cost of each.

10. Find two numbers such that if each is added to 5 times the other, the sums will be 43 and 47, respectively.

NOTE. If 3 is the tens' digit of a number, and 7 the units' digit, the number is 37 or $3 \times 10 + 7$. In like manner, if $x$ is the tens' digit and $y$ the units' digit, the number is $10x + y$.

**11.** The sum of the two digits of a number is 11, and if 63 is added to the number, the digits will be reversed. Find the number.

Let $x$ = the tens' digit,

and $y$ = the units' digit.

Then $10x + y$ = the number.

Hence, $x + y = 11$, (1)

and $10x + y + 63 = 10y + x$. (2)

From (2) $9x - 9y = -63$,

or $x - y = -7$. (3)

(1) + (3), $2x = 4$. $\therefore x = 2$.

(1) − (3), $2y = 18$. $\therefore y = 9$.

Therefore, the number is 29.

CHECK. $2 + 9 = 11$, and $29 + 63 = 92$.

**12.** The sum of the two digits of a number is 7, and when 45 is added to the number, the digits are reversed. Find the number.

**13.** A number, formed of two digits, is equal to 7 times the sum of the digits, and when 18 is subtracted from the number, the digits are reversed. Find the number.

**14.** There are two numbers such that the tens' digit of the first is the units' digit of the second and the tens' digit of the second is the units' digit of the first. Find the two numbers if their sum is 121 and their difference 45.

**15.** There is a certain number whose digits may be reversed by the addition of 27 to the number. Find the number if the sum of its digits is 9.

## III. INTEGERS OF HIGHER DEGREES

### DEFINITIONS

**51. Algebra**, like arithmetic, treats of numbers.

In arithmetic numbers are represented by figures; and in algebra numbers are represented by both figures and letters.

Hence, in algebra, letters are called numbers; and are always to be considered abstract numbers.

**52.** An indicated operation is to be regarded as an operation performed.

Thus, $a + b$ is the sum of $a$ and $b$ in the same sense that 13 is the sum of 10 and 3.

Therefore, any combination of figures and letters is a number; as $.6m$, $a + 3b$, $a \times b \times c$, $(a - n) \div y$.

Numbers are often called **quantities.**

**53.** The **factors** of a number are the numbers whose product is the given number.

Thus, 5, $a$, and $b$ are factors of $5 \times a \times b$, and $m$ and $a - b$ are factors of $m \times (a - b)$.

**54.** When no sign is written between numbers, the sign $\times$ is understood.

Thus, $5ab$ means $5 \times a \times b$, and $n(x + y)$ means $n \times (x + y)$.

**55.** A **power** is the result of multiplying a number by itself one or more times.

When a number is used *twice* as a factor, the product is the **second power** or the **square** of the number; when a number is used *three* times as a factor, the product is its **third power** or **cube**; when used *four* times as a factor, the product is its **fourth power**, etc.

Thus, 9 is the square of 3; 125 is the cube of 5; 16 is the fourth power of 2.

**56.** A power is indicated by a small figure or letter, called an **exponent**, written a little above and at the right of the number, showing how many times the number is taken as a factor.

Thus, $a^4$ means $a \times a \times a \times a$, and $(a+b)^3$ means $(a+b) \times (a+b) \times (a+b)$.

$a^2$ is read "$a$ square," or "$a$ second power"; $a^3$ is read "$a$ cube," or "$a$ third power"; $a^4$ is read "$a$ fourth," or "$a$ fourth power."

When no exponent is written, the exponent 1 is understood.

Thus, 2 is the first power of 2; $a = a^1$; $x = x^1$, etc.

A coefficient is the result of addition, and an exponent is the result of multiplication.

Thus, $a + a + a = 3a$, and $a \times a \times a = a^3$.

The number $5\,a^3b^5c^2d$ is usually read, "5 $a$ cube, $b$ fifth power, $c$ square, $d$," and its meaning is $5 \times a \times a \times a \times b \times b \times b \times b \times b \times c \times c \times d$.

Read, and write the meaning of:

| | | |
|---|---|---|
| 1. $7\,ab^2c^3$. | 2. $3\,a^2 + 5\,b^2$. | 3. $5\,a^2 + 3\,b - 4\,c$. |
| 4. $8\,x^3yz^4$. | 5. $7\,c^3 - 6\,x^2y$. | 6. $3\,x^3 - 5\,y^2 + 6\,z$. |
| 7. $5\,n^6r^5x$. | 8. $c^2x^4 + m^2y^3$. | 9. $8\,n^4 - 3\,nx - xy^2$. |

**57.** In combinations like the above the order of precedence of the indicated operations is:

*Power* operations are to be performed first; next, the *multiplications* and *divisions* (if any); last, the *additions* and *subtractions;* unless parentheses or other symbols of aggregation indicate that certain parts are to be considered as one part. See §§ 28 and 71.

Thus, $5 \times 2^3 - 4(6-2) + (6^2+4) \div 10 = 5 \times 8 - 4 \times 4 + 40 \div 10$
$= 40 - 16 + 4 = 28.$

**58.** A **monomial** is a number in which the figures and letters are not separated by the signs + or −.

Thus, $8a$, $7a^2b$, $5x^3y^4z^2$ are monomials.

**59.** A **polynomial** is a number in which two or more monomials are connected by the signs + or −.

The monomials are called the **terms** of the polynomial.

**60.** A **binomial** is a polynomial of two terms.

Thus, $6a^2b + b^3c$ is a binomial, the two terms being $6a^2b$ and $b^3c$.

**61.** A **trinomial** is a polynomial of three terms.

Thus, $5a - 6b + y$ is a trinomial, the three terms being $5a$, $-6b$, and $y$.

**62.** To **reduce** a number is to change its form or terms without changing its value or size.

**63.** To reduce a monomial is to reduce it to the least possible number of factors.

Thus, $5 \times 4 \times 3 = 20 \times 3 = 60$; and $a^3b^2a^2b = aaabbaab = a^5b^3$.

**64.** To reduce a polynomial is to reduce it to the least possible number of terms, expressed in their simplest forms.

$4 \times 3^2 - 5 \times 2^4 + 12 \div 4 = 4 \times 9 - 5 \times 16 + 3 = 36 - 80 + 3 = -41.$

$3x + 4y - 5x + 3y - 6 = 3x - 5x + 4y + 3y - 6 = -2x + 7y - 6.$

Reduce the following:

1. $6 \times 3 \times 5$.
2. $5 \times 4 + 3 \times 8 \times 2 - 3^2$.
3. $2^3 \times 2 \times 7$.
4. $4 \times 3^3 - 24 \div 6 - 5 \times 2^4$.
5. $5 \times 2^3 \times 3^2$.
6. $2 \times 3 \times 4 + (8 + 3^2 - 2) \div 5$.
7. $4 \times 5 \times 6^2$.
8. $8 + (9 + 5)(8 - 3) - 58$.
9. $2^4 \times 3^3 \times 10$.
10. $4^4 - 3^3 + 2^2 - 4 \times 3 \times 2$.
11. $4(10 - 3^2)$.
12. $(6 \times 8 - 4 \times 3^2) \div (5^2 - 3^2 - 2^2)$.
13. $5^2(8 - 6)^3$.
14. $(5^3 + 2^3) \div (5 + 2) - 4^2 - 3$.

**65. Finding the numerical values of algebraic numbers.**

If $a = 3$, $b = 2$, $c = 1$, $d = 0$, find the numbers represented by the following:

1. $a^2 + b$.
2. $5\,a^3 - 6\,a^2$.
3. $(a^2 - c^2) \div (a - c)$.
4. $5\,b - c^2$.
5. $7\,a(b^3 - c^3)$.
6. $(a^3 - c^3) \div (a - c)$.
7. $abc + d$.
8. $a^2 + 5\,bcd$.
9. $(a^4 - c^4) \div (a^2 + c^2)$.
10. $ac - bd$.
11. $(a^4 - c^4) \div b$.
12. $(a^6 - b^6) \div (a^3 - b^3)$.
13. $\dfrac{3\,a + c}{b} + \dfrac{6\,b + d}{a}$.
14. $\dfrac{4\,a + b}{c} \times \dfrac{5\,b + 2\,c}{a}$.
15. $\dfrac{7\,a - c}{b} - \dfrac{5\,b - c}{a}$.
16. $\dfrac{a^4 - c^4}{a - c} \div \dfrac{b^4 - d^4}{b - d}$.

Write the following as polynomials and then find their numerical values:

17. The sum of $a$ and five times $b$ square.
18. Two times $a$, less 4 times $c$ cube.
19. The product of $a$, $b$, $a - c$, and $b - d$.
20. The sum of $a$ cube and $\frac{1}{2}$ of the sum of $a$, $b$, and $c$.
21. Three times the sum of $a$ and $b$, increased by two times the difference between $c$ and $d$.
22. The sum of the cubes of $a$, $b$, and $d$.
23. Six more than $a$ square, divided by $c$, minus $b$ square.
24. The difference between the square of the sum of $a$ and $c$, and the sum of the squares of $a$ and $c$.
25. The difference between the cube of the sum of $a$ and $c$, and the sum of the cubes of $a$ and $c$.
26. Interpret $(a - c)^2 - (a^2 - c^2)$, and find its value.
27. Interpret $(a^4 - b^4) \div (a^2 - b^2)$, and find its value.

**66.** The terms of a polynomial preceded by the sign + are called **positive terms**, and the terms preceded by the sign − are called **negative terms**. The sign + before the first term is omitted.

**67. Like** or **similar terms** are terms that contain the same literal factors.

Thus, $+7a^2b$ and $-5a^2b$ are like terms, so are $8a^3(b-c)^2$ and $-7a^3(b-c)^2$.

**68. Unlike** or **dissimilar terms** are terms that do not contain the same literal factors.

Thus, $3a^2b$ and $5ab^2$, and $3x^2(x+y)^3$ and $5x^3(x+y)^2$ are unlike terms.

**69.** The **degree** of a term is the sum of the exponents of its literal factors.

Thus, $3x$, $4y$ are of the first degree; $3x^2$, $4ab$ are of the second degree; $3x^3$, $2a^2b$, $5abc$ are of the third degree.

**70.** A polynomial is said to be **homogeneous** when all its terms are of the same degree.

**71.** The **symbols of aggregation** are the parenthesis ( ), brace { }, bracket [ ], and vinculum $\overline{\quad}$. They are used to indicate that all the terms within them are to be treated as one part, or that the operations indicated within come before those without.

Thus, $8\times(5+2)=8\times7=56$; $12-\overline{5-2}^2=12-3^2=12-9=3$.

**72.** A polynomial is **arranged according to the powers** of some letter when the exponents of that letter, reckoning from left to right, either ascend or descend in the order of magnitude.

Thus, $x^3-3x^2y+3xy^2+y^3$ is arranged according to the *descending powers* of $x$ and the *ascending powers* of $y$.

## ADDITION

**73. Adding monomials.**

Find the sum of $8\,a^3b^2c$, $-9\,a^3b^2c$, and $5\,a^3b^2c$.

$$\begin{array}{r} +\,8\,a^3b^2c \\ -\,9\,a^3b^2c \\ +\,5\,a^3b^2c \\ \hline +\,4\,a^3b^2c \end{array}$$

The number $a^3b^2c$ may be represented by $y$. The required sum, then, is $8y - 9y + 5y$, or $4y$. Replacing $y$ by $a^3b^2c$, we have $4\,a^3b^2c$.

Consider $a^3b^2c$ a single quantity, and proceed as on page 20.

Find the sum of:

1. $6\,a^2c$, $8\,a^2c$, $-10\,a^2c$, $+9\,a^2c$, $-4\,a^2c$, $a^2c$.
2. $8\,xy^3$, $-9\,xy^3$, $4\,xy^3$, $-12\,xy^3$, $xy^3$, $6\,xy^3$.
3. $6\,x^2z^2$, $-12\,x^2z^2$, $x^2z^2$, $15\,x^2z^2$, $-8\,x^2z^2$, $3\,x^2z^2$.
4. $5\,ab^2c^3$, $-7\,ab^2c^3$, $-6\,ab^2c^3$, $+4\,ab^2c^3$, $ab^2c^3$.
5. $6\,(ab)^2$, $5\,(ab)^2$, $-4(ab)^2$, $-8\,(ab)^2$, $(ab)^2$, $3\,(ab)^2$.
6. $8\,(x-y)$, $-3(x-y)$, $11\,(x-y)$, $(x-y)$, $-5(x-y)$.

Reduce to the simplest form:

7. $9\,ab + 3\,ab - 7\,ab - ab + 6\,ab + 4\,ab - 8\,ab$.
8. $5\,a^2c - 8\,a^2c - 6\,a^2c + a^2c + 12\,a^2c - 3\,a^2c + 4\,a^2c$.
9. $14\,mn + 8\,mn - 3\,mn - mn + 5\,mn - 18\,mn$.
10. $16\,ax^2y - 9\,ax^2y + 4\,ax^2y - 7\,ax^2y - 3\,ax^2y - ax^2y$.
11. $7\,cyz^2 + 5\,cyz^2 - 8\,cyz^2 - 10\,cyz^2 + cyz^2$.
12. $5\,a^2bc^3 - 8\,a^2bc^3 - 10\,a^2bc^3 + 9\,a^2bc^3 + 7\,a^2bc^3$.
13. $-25\,x^3y^3 - 12\,x^3y^3 - 38\,x^3y^3 - 10\,x^3y^3 - x^3y^3$.
14. $6\,(a^2-b) + 16\,(a^2-b) - 25\,(a^2-b) + 14\,(a^2-b)$.
15. $4\,(a+b)^2 + 6\,(a+b)^2 - 9\,(a+b)^2 - 7\,(a+b)^2$.
16. $12\,(a+b+c) + 14\,(a+b+c) - 9\,(a+b+c)$.
17. $25\,(x+y+z) - 40\,(x+y+z) - 15\,(x+y+z)$.
18. $20\,(6x-7y) + 30\,(6x-7y) - 24\,(6x-7y)$.

**74. Adding polynomials.**

Find the sum of $3xy - 5y^2 - 8$, $6xy + 9y^2 + 6$, and $4xy - 8y^2 + 5$.

The addition is more conveniently made by writing the terms in columns, so that like terms shall stand in the same column.

$$\begin{array}{r} 3xy - 5y^2 - 8 \\ 6xy + 9y^2 + 6 \\ 4xy - 8y^2 + 5 \\ \hline 13xy - 4y^2 + 3 \end{array}$$

Find the sum of:

1. $6a^2b + 3ab - 5b^2$, $7a^2b - 8ab + 4b^2$, $a^2b + ab + b^2$.
2. $5a^2 + 6a - 7$, $9a^2 - 3a + 5$, $8 - 4a - 7a^2$.
3. $x^3 - x^2 + x - 1$, $3 - 2x - 4x^3$, $2x^2 + 3x - 2$.
4. $a^3 + a^2b + ab^2 + b^3$, $a^3 - a^2b + ab^2 - b^3$.
5. $2a^2 - 5ax + 6x^2$, $7ax - 8x^2 + 3a^2$, $9x^2 - 5a^2 - 2ax$.
6. $3x^3 - 4x + 7$, $2x^2 + 6x - 10$, $9x^3 - 4x^2 + 8$.
7. $6a^3 + 5a^2 + 3a$, $a^2 + a + 1$, $5 - 2a - 4a^2 - 6a^3$.
8. $2abcd + 3bx + m$, $2bx - 3m$, $6abcd - 8bx$.
9. $x^3 - 2 + 3x^2 - x$, $4x^2 + 8 - 2x^3$, $6x - 7x^2 - 6$, $x^3 - 5x$.
10. $ab + ac - 6$, $3ac + 8 - 3ab$, $10 - 5ab - 2ac$.
11. $15ax^3 - 6ay^3 + 5az^3$, $20ay^3 - 16ax^3 - 12az^3$, $24ax^3 - 8az^3 - 4ay^3$.

Reduce to the simplest form:

12. $5x^2 - 3xy + 8y^2 + 6xy - 8x^2 - 6y^2 + 9x^2 - 3xy + 5$.
13. $a^3 - 3a^2 + 6 + 5a - 2a^3 - 7 + 6a^2 + a^3 + 4 - 7a$.
14. $a^2c - ac^2 + 1 + x - 3ac^2 - 8 + 6a^2c + 7 - 5x + 4ac^2$.
15. $6(a + b) - 7(c - x) + 13 + 8(c - x) - 4(a + b) - 12$.
16. $7(a - x) - 6ax + 15 + 3(a - x) + 8ax - 10 - 6(a - x)$.
17. $6(x + y) - 5x^2y^2 - 2x^2 - 8(x + y) + 6x^2 - 3y^2 + 4(x + y) - 4y^2 - (x + y) - 3x^2$.

### SUBTRACTION

**75.** From $16\,x^2y - 5\,xy^2$ subtract $15\,x^2y + 4\,xy^2 - 6\,y$.

For convenience we write the subtrahend under the minuend so that like terms shall stand in the same column.

$$\begin{array}{r} 16\,x^2y - 5\,xy^2 \\ \underline{15\,x^2y + 4\,xy^2 - 6\,y} \\ x^2y - 9\,xy^2 + 6\,y \end{array}$$

Subtract as on page 25.

| | 1. | 2. | 3. | 4. |
|---|---|---|---|---|
| From | $9\,x^2y^3$ | $6\,a^3b$ | $-8\,c^4x^2$ | $-10\,m^2n$ |
| take | $5\,x^2y^3$ | $-5\,a^3b$ | $7\,c^4x^2$ | $-6\,m^2n$ |

**5.** From $5\,x^2 - 8\,x + 7$ subtract $3\,x^2 - 5\,x - 2$.

**6.** From $8\,a^2 + 3\,ab - 5\,b^2$ subtract $5\,a^2 + 7\,ab - 6\,b^2$.

**7.** From $6\,x^2y - 5\,xy^2 - 4\,xy + 5$ subtract $4\,x^2 + 3\,xy + 6$.

**8.** From $x^3 + 3\,x^2 - 6\,x + 4$ subtract $4\,x^2 - 6\,x - 3$.

**9.** From $4\,a + 3\,b + 2\,c$ subtract $2\,a + b - 2\,c$.

**10.** From $4\,a^3 + 6\,a^2 - 3\,a + 1$ take $3\,a^2 + 3\,a + 1$.

**11.** From $a^3 - 3\,a^2b + 3\,ab^2 - b^3$ take $a^2b + ab^2$.

**12.** From $a^2x^2 + 3\,ax + y$ take $a^2x^2 - 5\,ax - y$.

**13.** From $6\,ax + 7(a - b)$ take $3\,ax - 5(a - b)$.

**14.** From $8(a + b) - 7(x - y)$ take $3(a + b) + 2(x - y)$.

**15.** From $3\,a^2 + 3\,ab + 2\,ab^2 - 8$ take $2\,a^2 - ab + ab^2 - 7$.

**16.** The minuend is $6\,x^2 - 3\,xy + 8$ and the subtrahend is $-4\,x^2 - 5\,xy - 2$. Find the difference.

**17.** The minuend is $3\,a + b$ and the difference is $2\,a - c$. Find the subtrahend.

**18.** What number must be added to $3\,x^2 - 8\,x + 5$ to give $5\,x^2 - 11\,x - 3$?

**19.** Find $3\,a^2 + 6\,ab - x - (2\,a^2 - 2\,ab + y)$.

## MULTIPLICATION

**76. Multiplying a monomial by a monomial.**

Multiply $6\,a^3b^2c$ by $-3\,a^2bx$.

$6\,a^3b^2c \times -3\,a^2bx =$ $\quad +6\,a^3b^2c$

$6 \times -3\,a^3a^2b^2bcx =$ $\quad -3\,a^2bx$

$-18\,aaaaabbbcx = -18\,a^5b^3cx$ $\quad -18\,a^5b^3cx$

*To the product of the coefficients annex each literal factor with an exponent equal to the sum of the exponents of that factor in both the multiplicand and multiplier, observing the law of signs* (§ 31).

NOTE. The literal factors may be written in any order in the product, but for the sake of uniformity they are usually written in alphabetical order.

1. Multiply $+5\,a^2b$ by $+2\,ab$.
2. Multiply $+4\,x^3y^2$ by $-6\,x^5yz$.
3. Multiply $-8\,a^4bc^2$ by $+5\,ab^3x$.
4. Multiply $-6\,c^5d^3n$ by $-7\,d^4n^2x$.

Find the following:

5. $4\,a^2 \times 3\,ac$.
6. $9\,a^3b^4c \times 8\,a^2b^3d$.
7. $3\,x^3 \times -2\,x^2y$.
8. $7\,x^2y^5z \times -3\,x^7y^2z^5$.
9. $5\,a^4 \times -4\,a^2b$.
10. $5\,a\,(a-x)^2 \times 2\,a\,(a-x)$.
11. $-a^3d^2 \times -a^4d$.
12. $2\,x^2(a-b)^3 \times -6\,x^2(a-b)^4$.
13. $-x^2y \times 3\,x^3z$.
14. $3\,c^3d^4n \times -8\,cd^2n^3y$.
15. $8\,a^3b^2 \times 5\,a^2c^3$.
16. $4\,m^2\,(n+3)^2 \times 7\,m\,(n+3)$.
17. Multiply $-8\,a(x+y)^2$ by $-6\,a^2\,(x+y)^2$.
18. Multiply $+6\,a^2\,(a-b)^3$ by $-5\,b^2\,(a-b)^3$.
19. Multiply $-7\,ac\,(n+x)$ by $7\,a^2b\,(n+x)$.
20. Find $3\,x^2y \times 2\,x^3y^2z \times 5\,ay^4z$.
21. Find $2\,(a+x)^2 \times 3\,(a+x)^3 \times 4\,(a+x)^4$.

**77. Multiplying a polynomial by a monomial.**

Multiply $3a^2 - 4ab + b^2$ by $2ab$.

$$\begin{array}{r} 3a^2 - 4ab + b^2 \\ \underline{\qquad\qquad 2ab} \\ 6a^3b - 8a^2b^2 + 2ab^3 \end{array}$$

According to the distributive law of multiplication (§ 33), we multiply each term of the multiplicand by the multiplier, and take the algebraic sum of the products.

Multiply:

1. $x^2 + 8x$ by $5x$.
2. $3x^2y - 6xy^2 + 8y^3$ by $-3xy^2$.
3. $a^3 - 5a^2$ by $-2a$.
4. $4a^2b^3 + 5ab^4 - 2b^5$ by $2a^2b^2$.
5. $3x - 4y$ by $-3x$.
6. $3n^4 - 4n^3r + n^2r^2$ by $4n^2r^4$.
7. $4x - 5y$ by $-4y$.
8. $5c^3 - 2c^2d - 9cd^2$ by $-c^3d^3$.
9. $x^3y + y^2$ by $x^2y$.
10. $6n^4 - 4n^2 + 8$ by $7a^2n^3$.

Find the value of:

11. $(x^3 - 4xy) \times -x^2$.
12. $(4x^2 - 3xy - 6y^2) \times 2x^2y$.
13. $(x^4 + 3x^2y) \times -x^3$.
14. $(x^2y^2 - 2xy + 4) \times -4xy$.
15. $(a^5 - 5a^3) \times 2a^2$.
16. $(a^3 + 2a^2b + 6b^3) \times 3ab^3$.

Find the product of:

17. $-x^3 - 4x^2y + 3y^2$ and $-x^2$.
18. $3a^2b - 5ab^2 - 8b^3$ and $2a^2b^3$.
19. $a^2 - 8ab^2 + 6b^4$ and $5ab^2$.
20. $ab + ax + xy$ and $abxy$.
21. $x^4y^2 - x^2y^4 + xy^6$ and $xy^3$.
22. $6m^2 - 8mn - n^2$ and $2mn$.
23. $3a^2x - 5x^2y + 2y^2$ and $-4ay^2$.
24. $x^4 + x^3 + x^2 + x$ and $-5x^2$.
25. $6a^2 - 3c^2 + 2e^2$ and $-ace$.
26. $5(a + x)^2 + 3(a + x)^3$ and $4(a + x)$.
27. $3c(a - b) - 4a(a - b)^2$ and $-2ac(a - b)^2$.

**78. Multiplying a polynomial by a polynomial.**

1. Multiply $m+n$ by $m+n$.

$$\begin{array}{rl} & m+n \\ & m+n \\ \hline m \text{ times } m+n = & m^2+mn \\ n \text{ times } m+n = & \quad mn+n^2 \\ \hline (m+n) \text{ times } m+n = & m^2+2mn+n^2 \end{array}$$

*Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

2. Multiply $2a^2-3ab+4b^2$ by $3a-2b$.

$$\begin{array}{r} 2a^2-3ab+4b^2 \\ 3a-2b \\ \hline 6a^3-9a^2b+12ab^2 \\ \quad -4a^2b+6ab^2-8b^3 \\ \hline 6a^3-13a^2b+18ab^2-8b^3 \end{array}$$

Multiply:

1. $a+b$ by $a+b$.
2. $x^2+2y$ by $x^2-2y$.
3. $a+b$ by $a-b$.
4. $x^2-3xy$ by $x+4y$.
5. $2a+c$ by $3a-c$.
6. $x^3+x^2$ by $x^2+x$.
7. $4x-2y$ by $4x+3y$.
8. $x^2+x+1$ by $x-1$.

Find the value of:

9. $(x^3+2x^2+4x+8)\times(x-2)$.
10. $(y^3-2y^2+4y-8)\times(y+2)$.
11. $(3x^2y-5xy^2+8y^3)\times(2xy-y^2)$.
12. $(a^3-3a^2b+3ab^2-b^3)\times(a^2+2ab+b^2)$.

Find the product of:

13. $a^2+b^2$ and $a^2-b^2$.
14. $a+b+c$ and $a+b-c$.
15. $a^3+a^2$ and $a^3-a^2$.
16. $2a+x-y$ and $a-2x$.
17. $3x^2+1$ and $3x^2-1$.
18. $3x+2y-6$ and $2x-5y$.
19. $5+x^3$ and $5-x^3$.
20. $x^4+4x^2+16$ and $x^2-4$.

**79. Finding the square of the sum of two numbers.**

Find $(a+b)^2$; $(2x+3)^2$.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{r} 2x+3 \\ 2x+3 \\ \hline 4x^2+6x \\ \quad +6x+9 \\ \hline 4x^2+12x+9 \end{array}$$

*The square of the sum of two numbers is the sum of their squares plus twice their product.*

Write the values of the following by the above rule:

1. $(a+c)(a+c)$.
2. $(x+1)^2$.
3. $(2x+4)^2$.
4. $(b+x)(b+x)$.
5. $(a+6)^2$.
6. $(2x+3)^2$.
7. $(x+y)(x+y)$.
8. $(y+7)^2$.
9. $(3y+2)^2$.
10. $(x+2)(x+2)$.
11. $(a+n)^2$.
12. $(4a+b)^2$.
13. $(x+5)(x+5)$.
14. $(c+y)^2$.
15. $(nx+d)^2$.

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16. Square of $2x+3y$.
17. $(5x^2y+2xy^2)^2$.
18. Square of $3x+4y$.
19. $(3ab+5ac)^2$.
20. Square of $x^2+6x$.
21. $(2cx+3x^2y)^2$.
22. Square of $a^2+2a$.
23. $(4a^2+5abc)^2$.
24. Square of $5a+3b$.
25. $(3a^2x+10ay)^2$.
26. Square of 24 or $(20+4)^2$.
27. $32^2$ or $(30+2)^2$.
28. $45^2$ or $(40+5)^2$.
29. $18^2$ or $(10+8)^2$.
30. $54^2$ or $(50+4)^2$.
31. $62^2$ or $(60+2)^2$.
32. $83^2$ or $(80+3)^2$.

**80. Finding the square of the difference of two numbers.**

Find $(a-b)^2$; $(3x^2-2xy)^2$.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array} \qquad \begin{array}{r} 3x^2-2xy \\ 3x^2-2xy \\ \hline 9x^4-6x^3y \\ -6x^3y+4x^2y^2 \\ \hline 9x^4-12x^3y+4x^2y^2 \end{array}$$

*The square of the difference of two numbers is the sum of their squares minus twice their product.*

Write the values of the following by the above rule:

1. $(a-x)(a-x)$.
2. $(x-3)^2$.
3. $(2x-5)^2$.
4. $(b-y)(b-y)$.
5. $(c-7)^2$.
6. $(4x-3)^2$.
7. $(n-r)(n-r)$.
8. $(5-y)^2$.
9. $(5y-6)^2$.
10. $(x-6)(x-6)$.
11. $(a-n)^2$.
12. $(3a-b)^2$.
13. $(x-8)(x-8)$.
14. $(x-9)^2$.
15. $(8-3x)^2$.

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16. Square of $3x-2y$.
17. $(4a^2b-3ab^2)^2$.
18. Square of $4a-3b$.
19. $(5xy^2-10yz)^2$.
20. Square of $x^2-5x$.
21. $(5ab-6acx)^2$.
22. Square of $a^2-3a$.
23. $(3ax-2x^2y)^2$.
24. Square of $5a-3b$.
25. $(a^2x-12ax^2)^2$.
26. Square of 28 or $(30-2)^2$.
27. $16^2$ or $(20-4)^2$.
28. $59^2$ or $(60-1)^2$.
29. $37^2$ or $(40-3)^2$.
30. $88^2$ or $(90-2)^2$.
31. $48^2$ or $(50-2)^2$.
32. $99^2$ or $(100-1)^2$.

**81. Finding the product of the sum and difference of two numbers.**

Find $(a+b)(a-b)$; $(2x^2+3xy)(2x^2-3xy)$.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2 \quad -b^2 \end{array} \qquad \begin{array}{r} 2x^2+3xy \\ 2x^2-3xy \\ \hline 4x^4+6x^3y \\ -6x^3y-9x^2y^2 \\ \hline 4x^4 \quad -9x^2y^2 \end{array}$$

*The product of the sum and difference of two numbers is the difference of their squares.*

Find the values of the following by the above rule:

1. $(x+y)(x-y)$.
2. $(3x+2y)(3x-2y)$.
3. $(x+5)(x-5)$.
4. $(5a+4b)(5a-4b)$.
5. $(a+7)(a-7)$.
6. $(nx+5y)(nx-5y)$.
7. $(6+b)(6-b)$.
8. $(x^2+6x)(x^2-6x)$.
9. $(8-c)(8+c)$.
10. $(9a+my)(9a-my)$.

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11. $(x^2+8)(x^2-8)$.
12. $(3x^2y+4xy^2)(3x^2y-4xy^2)$.
13. $(a^3+7)(a^3-7)$.
14. $(5ac^3+3bc^2)(5ac^3-3bc^2)$.
15. $(5+a^4)(5-a^4)$.
16. $(4n^2x-8ny^3)(4n^2x+8ny^3)$.
17. $(6-c^5)(6+c^5)$.
18. $(12x^4-10c^5)(12x^4+10c^5)$.

Find the following in like manner and check by arithmetic:

19. $(10+6)(10-6)$.
20. $(30+7)(30-7)$.
21. $(12+4)(12-4)$.
22. $(70+3)(70-3)$.
23. $(20+5)(20-5)$.
24. $(80+4)(80-4)$.
25. $(15+4)(15-4)$.
26. $(100+8)(100-8)$.
27. $(50+6)(50-6)$.
28. $(100+3)(100-3)$.

**82. Finding the product of two binomials of the form $(x+a)(x+b)$.**

Find $(x+6)(x+3)$; $(x-7)(x+2)$; $(x-4)(x-5)$.

$$\begin{array}{lll} x+6 & x-7 & x-4 \\ \underline{x+3} & \underline{x+2} & \underline{x-5} \\ x^2+6x & x^2-7x & x^2-4x \\ \underline{\quad +3x+18} & \underline{\quad +2x-14} & \underline{\quad -5x+20} \\ x^2+9x+18 & x^2-5x-14 & x^2-9x+20 \end{array}$$

How many terms has each of these products?

What is the first term of each product?

In the first product the coefficient of $x$ $(+9)$ is the algebraic sum of the second terms $(+6, +3)$ of the binomials. See whether this is true of the other products.

In the first product the last term $(+18)$ is the product of the second terms $(+6, +3)$ of the binomials. See whether this is true of the other products.

Hence, to multiply $x+8$ by $x-5$, we take for the first term $x^2$, for the second, $(+8$ plus $-5)\,x$, or $+3x$, and for the third, $+8 \times -5$, $-40$, which gives $x^2+3x-40$.

Find, in this manner, the following:

1. $(x+4)(x+7)$.
2. $(x+10)(x-12)$.
3. $(x-3)(x+8)$.
4. $(x-2a)(x+4a)$.
5. $(x+5)(x-9)$.
6. $(x-15)(x-20)$.
7. $(x-3)(x-7)$.
8. $(x+5n)(x-7n)$.
9. $(x+8)(x-6)$.
10. $(x+30)(x-20)$.
11. $(x-9)(x+4)$.
12. $(x+7c)(x-4c)$.
13. $(x+5)(x+7)$.
14. $(x-9a)(x+6a)$.
15. $(x-9)(x-8)$.
16. $(x+6b)(x+7b)$.
17. $(x+6)(x-7)$.
18. $(x-9c)(x-5c)$.

## DIVISION

**83. Dividing a monomial by a monomial.**

Divide $-15\,a^5bc^3$ by $5\,a^2bc$.

$-15 \div 5 = -3$
$a^5 \div a^2 = a^3$, because $a^3 \times a^2 = a^5$
$b \div b = 1$, because $1 \times b = b$
$c^3 \div c = c^2$, because $c^2 \times c = c^3$.

$5\,a^2bc)\overline{-15\,a^5bc^3}$
$\quad -3\,a^3c^2$

Hence, the quotient is $-3\,a^3c^2$.

*Divide the coefficient of the dividend by the coefficient of the divisor, observing the law of signs. To this quotient annex each literal factor of the dividend with an exponent equal to the exponent of that factor in the dividend minus its exponent in the divisor.*

1. Divide $36\,a^3b^2c$ by $9\,abc$.
2. Divide $42\,a^4b^2c^3x$ by $-6\,a^2bc^2$.
3. Divide $-20\,x^4yz$ by $4\,x^3y$.
4. Divide $-35\,c^5x^3y$ by $-5\,cx^3$.

Find the following:

5. $12\,a^3c \div 3\,ac$.
6. $-21\,x^9y^7z^6 \div 7\,x^2y^5z$.
7. $24\,a^4c^2x \div 8\,a^3c$.
8. $72\,a^5b^7cd \div 9\,a^3b^4c$.
9. $28\,a^6b^3 \div -4\,a^3b$.
10. $-8\,a^3b^4c^2x \div -4\,abc^2$.
11. $3\,a^7b^4c^2x \div a^4b^4c$.
12. $48\,c^3x^4y^2z \div 6\,x^3yz$.
13. Divide $18\,a^2(a-b)^4$ by $6\,a(a-b)^2$.
14. Divide $27\,c^3(a-x)^7$ by $9\,c(a-x)^4$.
15. Divide $m^3(x+2)^8$ by $m^3(x+2)^5$.
16. Divide $8\,a^6(c-d)^9$ by $2\,a^3(c-d)^3$.
17. Divide $7\,a^5(x-7)^6$ by $7\,a^2(x-7)^2$.

**84. Dividing a polynomial by a monomial.**

Divide $15\,a^3b^2x^4 - 18\,a^4b^3x^3 + 27\,a^2b^5x^6$ by $3\,a^2b^2x^3$.

$$3\,a^2b^2x^3)\overline{15\,a^3b^2x^4 - 18\,a^4b^3x^3 + 27\,a^2b^5x^6}$$
$$5\,ax \quad - \quad 6\,a^2b \quad + \quad 9\,b^3x^3$$

*According to* § 37, *we divide each term of the dividend by the divisor and take the algebraic sum of the quotients.*

Divide :

1. $x^2 + 6\,x$ by $x$.
2. $3\,x^3y - 6\,x^2y^2 + 9\,xy^3$ by $3\,xy$.
3. $a^3 - 8\,a^2b$ by $a^2$.
4. $8\,a^4b^5 + 4\,a^3b^6 - 2\,a^2b^7$ by $2\,a^2b^2$.
5. $4\,y^3 - 6\,y^2z$ by $2\,y^2$.
6. $5\,n^2x - 3\,n^3x^3 - n^4x^2$ by $-n^2x$.
7. $a^2n - an^2$ by $-an$.
8. $x^2y^2 - x^3y - xy^3$ by $xy$.

Find the value of :

9. $(27\,x^5 - 45\,x^3) \div 9\,x^2$.
10. $(-x^6 + 2\,x^5 - x^4) \div -x^4$.
11. $(24\,y^6 - 6\,y^3) \div -6\,y^3$.
12. $(a^2xy^3 - ax^3y^2 + a^3x^2y) \div axy$.
13. $(6\,x^3 - 2\,x^5) \div -2\,x^3$.
14. $(54\,a^5b^2 - 135\,a^4b^7) \div 9\,a^4b^2$.

Find the quotient of :

15. $x^5 + 4\,x^4y - 3\,x^2y^2$ by $-x^2$.
16. $xy - x^2y^2 - 5\,x^3y^3$ by $-xy$.
17. $6\,x^5y^2 - 9\,x^4y^3 + 3\,x^2y^4$ by $3\,x^2y^2$.
18. $3\,x^3 - 6\,x^5 + 9\,x^7 - 12\,x^9$ by $3\,x^2$.
19. $42\,x^4 - 14\,x^3 + 28\,x^2 - 7\,x$ by $7\,x$.
20. $5(a + x)^4 - 10(a + x)^3$ by $5(a + x)^2$.
21. $c^2(a - b)^2 + c^4(a - b)^5$ by $c^2(a - b)^2$.
22. $9\,x^3(a + 2)^3 - 6\,x^2(a + 2)^6$ by $3\,x^2(a + 2)^3$.
23. $15\,a(x - 7)^8 - 10\,a(x - 7)^6$ by $5\,a(x - 7)^4$.

## 85. Dividing a polynomial by a polynomial.

1. Divide $9x + 20 + x^2$ by $5 + x$.

$$\begin{array}{l|l} x^2 + 9x + 20 & x + 5 \\ \underline{x^2 + 5x\phantom{ + 20}} & x + 4 \\ \quad 4x + 20 & \\ \quad \underline{4x + 20} & \end{array}$$

Arrange both dividend and divisor in descending powers of $x$.

The first term of the dividend, $x^2$, divided by the first term of the divisor, $x$, gives $x$, the first term of the quotient.

Multiplying the divisor, $x + 5$, by $x$, and subtracting the product, $x^2 + 5x$, from the dividend, leaves the remainder $4x + 20$.

The first term of this remainder, $4x$, divided by the first term of the divisor, $x$, gives 4, the second term of the quotient.

Multiplying the divisor, $x + 5$, by 4, and subtracting the product, $4x + 20$, from the remainder, leaves 0. Therefore, $x + 4$ is the quotient.

*Write the divisor at the right of the dividend, arranging the terms of each according to the ascending or descending powers of one of the letters.*

*Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.*

*Multiply the divisor by this term of the quotient, subtract the product from the dividend, and to the remainder annex one or more terms for a new dividend.*

*Divide the new dividend as before, and continue to divide until there is no remainder, or until the first term of the divisor is not contained in the first term of the dividend.*

2. Divide $6x^2 - 7x - 20$ by $3x + 4$.

$$\begin{array}{l|l} 6x^2 - 7x - 20 & 3x + 4 \\ \underline{6x^2 + 8x\phantom{ - 20}} & 2x - 5 \\ \quad -15x - 20 & \\ \quad \underline{-15x - 20} & \end{array}$$

*Ans.* $2x - 5$.

3. Divide $x^3 + 27$ by $x + 3$.

$$
\begin{array}{rl}
x^3 + 27 & \mid x + 3 \\
\underline{x^3 + 3x^2} & \quad x^2 - 3x + 9 \\
-3x^2 + 27 & \\
\underline{-3x^2 - 9x} & \\
+9x + 27 & \\
\underline{+9x + 27} & \qquad \textit{Ans.}\ x^2 - 3x + 9.
\end{array}
$$

NOTE. Arrange each remainder in the same order as the dividend and divisor.

Divide:

1. $x^2 - 10x + 21$ by $x - 3$.
2. $x^2 - 4$ by $x + 2$.
3. $x^2 - 12x + 32$ by $x - 4$.
4. $x^2 - 9$ by $x - 3$.
5. $x^2 - 11x - 42$ by $x + 3$.
6. $x^2 - a^2$ by $x + a$.
7. $x^2 + 10x - 75$ by $x - 5$.
8. $x^3 + 8$ by $x + 2$.
9. $x^2 + 15x - 76$ by $x - 4$.
10. $x^3 - 8$ by $x - 2$.
11. $x^2 - 20x - 69$ by $x + 3$.
12. $x^4 - 1$ by $x - 1$.

Find the value of the following:

13. $(6x^2 - 13x - 28) \div (2x - 7)$.
14. $(6x^2 + 35x + 25) \div (3x + 5)$.
15. $(8x^2 + 38x + 42) \div (2x + 6)$.
16. $(8x^2 + 10x - 33) \div (2x - 3)$.
17. $(9x^2 - 12x - 60) \div (3x + 6)$.
18. $(a^2x^2 - 2ax - 24) \div (ax + 4)$.
19. $(n^2x^2 + 2nx - 35) \div (nx - 5)$.
20. $(a^4x^2 + 3a^2x - 40) \div (a^2x + 8)$.
21. $(n^2x^2 - 3nx - 18) \div (nx - 6)$.
22. $(a^4x^4 - 4a^2x^2 - 21) \div (a^2x^2 - 7)$.

Divide $36 + x^4 - 13x^2$ by $6 + x^2 + 5x$.

Arrange the terms in descending powers of $x$.

$$\begin{array}{rl} x^4 - 13x^2 + 36 & \big| \; x^2 + 5x + 6 \\ \underline{x^4 + 5x^3 + 6x^2} & \quad x^2 - 5x + 6 \\ -5x^3 - 19x^2 + 36 & \\ \underline{-5x^3 - 25x^2 - 30x} & \\ +6x^2 + 30x + 36 & \\ \underline{+6x^2 + 30x + 36} & \end{array}$$

Divide:

1. $2x^3 - x^2 + 3x - 9$ by $2x - 3$.
2. $x^3 - 5x^2 - x + 14$ by $x - 2$.
3. $9x^3 + 14x + 5$ by $3x^2 - x + 5$.
4. $6x^3 - 13x^2 + 4$ by $2x^2 - 3x - 2$.
5. $x^4 + 4x^2 + 16$ by $x^2 + 2x + 4$.
6. $8a^3 + 12a^2 + 6a + 1$ by $2a + 1$.
7. $a^4 + a^2x^2 + x^4$ by $a^2 + ax + x^2$.
8. $4a^4 - 9a^2 + 6a - 1$ by $2a^2 + 3a - 1$.
9. $a^6 - 6a^4 + 5a^2 - 1$ by $a^3 + 2a^2 - a - 1$.
10. $a^2 - b^2 - c^2 - 2bc$ by $a - b - c$.
11. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
12. $a^5 - 17a^3 + 12a^2 + 52a - 48$ by $a^2 + a - 2$.
13. $x^3 + 64$ by $x + 4$.
14. $64 + 27a^3$ by $4 + 3a$.
15. $x^5 + 32$ by $x + 2$.
16. $16 - 81x^4$ by $2 + 3a$.
17. $x^4 - 81$ by $x - 3$.
18. $8 - 125c^3$ by $2 - 5c$.
19. $16 - a^4$ by $2 - a$.
20. $216x^3 - 1$ by $6x - 1$.

## FACTORING

**86. Factoring** is the process of expressing a number in terms of its prime factors.

**87.** A **prime number** is one that has no factors except itself and 1.

Factors that are prime numbers are called **prime factors.**

Letters, for the present, are supposed to represent integers; and, until some numerical value is assigned to them, they are regarded and treated as prime numbers.

A factor of two or more numbers is called a **common factor** of them.

**88.** Monomials, except when their coefficients are not prime numbers, are expressed in terms of their factors.

Thus, the prime factors of $35\,a^2bc^3$ are 5, 7, $a$, $a$, $b$, $c$, $c$, $c$.

Hence, the factors of monomials may be found by inspection.

**89. Polynomials all of whose terms have a common factor.**

1. Factor $6\,a^2c^3 - 9\,a^3c$.

$$3\,a^2c)\,6\,a^2c^3 - 9\,a^3c$$
$$2\,c^2 - 3\,a$$

The greatest factor common to both terms is $3\,a^2c$. Dividing the given polynomial by $3\,a^2c$ gives the quotient $2\,c^2 - 3\,a$.

$$\therefore\ 6\,a^2c^3 - 9\,a^3c = 3\,a^2c(2\,c^2 - 3\,a).$$

2. Factor $3\,a^2(c+d) + 4\,ax\,(c+d)$.

The greatest factor common to both terms is $a(c+d)$. Dividing the given polynomial by this common factor gives the quotient $3\,a + 4\,x$.

$$\therefore\ 3\,a^2\,(c+d) + 4\,ax\,(c+d) = a\,(c+d)\,(3\,a + 4\,x).$$

Factor:

| | | |
|---|---|---|
| **1.** $2a + 2c$. | **2.** $a^2 - ac$. | **3.** $6a^2c - 8ac^2$. |
| **4.** $ab + bc$. | **5.** $ax + x^2$. | **6.** $4a^3 + 2a^2x$. |
| **7.** $3c + 18$. | **8.** $3a^2 - a$. | **9.** $nb^2x - n^2by$. |
| **10.** $a + abc$. | **11.** $x^3 + ax^2$. | **12.** $c^3x^2 + c^2x^3$. |

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| | |
|---|---|
| **13.** $4a^3 + 8a^2 + 12a$. | **14.** $a^2c - a^2c^2 + a^3c^3 + ac^4$. |
| **15.** $3x^2 + 9x^3 - 6x^4$. | **16.** $6x^3y + 12xy^2 - 18x^2y^2$. |
| **17.** $2n^3 - 4n^4 + 8n^5$. | **18.** $24m^4 - 40m^3 - 56m^6$. |
| **19.** $7x^2 - nx^4 - nx$. | **20.** $a^4b - a^3b^2 + a^2b^3 - ab^4$. |

---

| | |
|---|---|
| **21.** $a(c+b) + n(c+b)$. | **22.** $2a(x+2) + 4c(x+2)$. |
| **23.** $a(b-x) + c(b-x)$. | **24.** $3c(a-x) + 5b(a-x)$. |
| **25.** $n(x+2) - a(x+2)$. | **26.** $4a(n+3) - 6b(n+3)$. |
| **27.** $a(x-y) - b(x-y)$. | **28.** $9c(x-4) - 6x(x-4)$. |

**29.** $a(x+y) - b(x+y) + c(x+y)$.

**30.** $x(a-b) + y(a-b) - 3(a-b)$.

**90. Polynomials some of whose terms have a common factor.**

1. Factor $ax + ay + bx + by$.

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (a + b)(x + y). \end{aligned}$$

We group the terms $ax$ and $ay$ because they have the common factor $a$, and we group $bx$ and $by$ because they have the common factor $b$.

We might also have grouped them thus:

$$\begin{aligned} ax + ay + bx + by &= (ax + bx) + (ay + by) \\ &= x(a + b) + y(a + b) \\ &= (x + y)(a + b). \end{aligned}$$

2. Factor $ax + ay - bx - by$.

$$\begin{aligned} ax + ay - bx - by &= (ax + ay) - (bx + by) \\ &= a(x + y) - b(x + y) \\ &= (a-b)(x+y). \end{aligned}$$

The signs of the last two terms, $-bx - by$, must be changed when put within a parenthesis preceded by the sign $-$.

3. Factor $x^3 - x^2 + 3x - 3$.

$$\begin{aligned} x^3 - x^2 + 3x - 3 &= (x^3 - x^2) + (3x - 3) \\ &= x^2(x - 1) + 3(x - 1) \\ &= (x^2 + 3)(x - 1). \end{aligned}$$

4. Factor $6x^3 - 8x^2 - 15x + 20$.

$$\begin{aligned} 6x^3 - 8x^2 - 15x + 20 &= 6x^3 - 15x - 8x^2 + 20 \\ &= (6x^3 - 15x) - (8x^2 - 20) \\ &= 3x(2x^2 - 5) - 4(2x^2 - 5) \\ &= (3x - 4)(2x^2 - 5). \end{aligned}$$

Factor the following:

1. $ax - ay + bx - by$.
2. $x^3 + 2x^2 + 6x + 12$.
3. $ax - ay - bx + by$.
4. $x^3 - 4x^2 + 5x - 20$.
5. $ac - 3a + 2c - 6$.
6. $x^3 - 3x^2 - 6x + 18$.
7. $ax + 4a - 2x - 8$.
8. $x^3 + 4x^2 - 3x - 12$.
9. $nx - 3n - 3x + 9$.
10. $x^3 - 5x^2 - 7x + 35$.
11. $x^2 - ax + nx - an$.
12. $x^4 - 6x^3 + 5x - 30$.
13. $x^2 + 6x + 8$.

$$\begin{aligned} x^2 + 6x + 8 &= x^2 + 2x + 4x + 8 \\ &= x(x + 2) + 4(x + 2) = (x + 4)(x + 2). \end{aligned}$$

Separate the middle term, $6x$, into two parts, $2x$ and $4x$, such that the product of their coefficients shall be the last term.

14. $x^2 + 7x + 12$.
15. $x^2 + 9x + 20$.
16. $x^2 + 11x + 30$.
17. $x^2 - 9x + 14$.
18. $x^2 - 8x + 12$.
19. $x^2 - 13x + 42$.
20. $x^2 + 8x + 15$.
21. $x^2 - 7x + 10$.
22. $x^2 - 15x + 56$.

### SQUARE ROOT AND CUBE ROOT

**91.** If a number is the product of two equal factors, one of the factors is called the **square root** of the number.

Thus, since $x^2 = x \times x$, $x$ is the square root of $x^2$. Again, since $(a + x)^2 = (a + x)(a + x)$, $a + x$ is the square root of $(a + x)^2$.

1. Name the square roots of the following: 9, 25, 64, $a^2$, $x^4$, $4x^2$, $(a + b)^2$, $(a - c)^4$.

Since $6a^4b^3c^2x \times 6a^4b^3c^2x = 36a^8b^6c^4x^2$, $6a^4b^2c^2x$ is the square root of $36a^8b^6c^4x^2$.

*To extract the square root of a monomial, extract the square root of the coefficient, and divide the exponent of each literal factor by 2.*

Find the square root of:

2. $9x^4$ $49a^6b^2$ $a^8x^{10}$ $16(a + b)^4$ $64a^8(a + b)^6$
3. $16a^8$ $81m^4n^8$ $n^6y^8$ $25(x + y)^6$ $36b^4(x + y)^4$

**92.** If a number is the product of three equal factors, one of the factors is called the **cube root** of the number.

Thus, since $a^3 = a \times a \times a$, $a$ is the cube root of $a^3$. Again, since $(a + b)^3 = (a + b)(a + b)(a + b)$, $a + b$ is the cube root of $(a + b)^3$.

4. Name the cube roots of: 8; 27; 125; $a^3$; $a^6$; $64x^9$; $(a - x)^3$; $(a + c)^6$.

Since $27a^6b^3 = 3a^2b \times 3a^2b \times 3a^2b$, $3a^2b$ is the cube root of $27a^6b^3$.

*To extract the cube root of a monomial, extract the cube root of the coefficient, and divide the exponent of each literal factor by 3.*

Find the cube root of:

5. $27x^6$ $216a^9b^3$ $a^{12}x^9$ $64(a + b)^6$ $125a^6(x - 2)^1$
6. $8a^9$ $343x^6y^9$ $x^6y^{12}$ $512(x + y)^3$ $729x^3(a - b)^{62}$

**93. Factoring the difference of two squares.**

Since $a^2 - b^2$ divided by $a + b$ gives $a - b$, the factors of $a^2 - b^2$ are $a + b$ and $a - b$.

*To find the factors of the difference of two squares, take the sum of the square roots of the terms for one factor, and the difference of the square roots of the terms for the other factor.*

1. Factor $9\,a^4 - 25\,x^6$.

The square root of $9\,a^4$ is $3\,a^2$, and the square root of $25\,x^6$ is $5\,x^3$. Therefore, by the rule,

$$9\,a^4 - 25\,x^6 = (3\,a^2 + 5\,x^3)(3\,a^2 - 5\,x^3).$$

2. Factor $(a + b)^2 - c^2$.

The square root of $(a + b)^2$ is $a + b$, and the square root of $c^2$ is $c$.

$$\therefore \quad (a + b)^2 - c^2 = (a + b + c)(a + b - c).$$

Factor:

| | | |
|---|---|---|
| 1. $x^4 - a^2$. | 2. $x^2 - 64$. | 3. $25\,a^2 - 16\,b^2$. |
| 4. $a^2 - c^8$. | 5. $y^2 - 25$. | 6. $n^2x^4 - 49\,y^2$. |
| 7. $a^4 - 9$. | 8. $z^6 - 36$. | 9. $c^2x^4y^6 - m^2$. |
| 10. $1 - m^2$. | 11. $9 - c^2x^2$. | 12. $a^4c^2 - a^2c^4$. |

13. $x^4 - 81$.

$$x^4 - 81 = (x^2 - 9)(x^2 + 9) = (x^2 + 9)(x + 3)(x - 3).$$

| | | |
|---|---|---|
| 14. $a^4 - 16$. | 15. $x^4 - 625$. | 16. $x^8 - 256$. |
| 17. $1 - 144\,a^6b^2c^8$. | 18. $(x - y)^2 - 4\,z^2$. | |
| 19. $49\,a^2 - 64\,a^6b^6$. | 20. $(a + b)^2 - 9\,c^2$. | |
| 21. $16\,c^4 - 81\,c^2x^8$. | 22. $(a - x)^2 - n^2x^2$. | |
| 23. $(2\,x + 3)^2 - 25$. | 24. $c^2 - (a + 2\,b)^2$. | |
| 25. $(3\,x - 2)^2 - 4\,x^2$. | 26. $a^2 - (b - 3\,c)^2$. | |
| 27. $(4\,x + 10)^2 - 1$. | 28. $1 - 4(a + x)^2$. | |

**94. Factoring the difference of two cubes.**

Since $a^3 - b^3$ divided by $a - b$ gives $a^2 + ab + b^2$, the factors of $a^3 - b^3$ are $a - b$ and $a^2 + ab + b^2$.

*To find the factors of the difference of two cubes, take the difference of the cube roots of the terms for one factor, and the sum of the squares of the cube roots of the terms plus their product for the other factor.*

Factor $8x^6 - n^3x^9$.

The cube root of $8x^6$ is $2x^2$, and the cube root of $n^3x^9$ is $nx^3$. Therefore, by the rule,

$$8x^6 - n^3x^9 = (2x^2 - nx^3)(4x^4 + 2nx^5 + n^2x^6).$$

The second factor may be found by dividing $8x^6 - n^3x^9$ by the first factor.

Factor:

| | | |
|---|---|---|
| **1.** $x^6 - a^3$. | **2.** $x^3 - 64$. | **3.** $27x^3 - 512y^3$. |
| **4.** $a^9 - b^6$. | **5.** $27 - n^6$. | **6.** $64a^3 - 729b^3$. |
| **7.** $c^3 - 8$. | **8.** $m^3 - x^9$. | **9.** $8x^6y^3 - 125z^3$. |
| **10.** $1 - y^6$. | **11.** $a^3c^6 - 1$. | **12.** $216 - 343c^3$. |

**95. Factoring the sum of two cubes.**

Since $a^3 + b^3$ divided by $a + b$ gives $a^2 - ab + b^2$, the factors of $a^3 + b^3$ are $a + b$ and $a^2 - ab + b^2$.

*To find the factors of the sum of two cubes, take the sum of the cube roots of the terms for one factor, and the sum of the squares of the cube roots of the terms minus their product for the other factor.*

Factor $27 + 8x^9$.

The cube root of 27 is 3 and the cube root of $8x^9$ is $2x^3$. Therefore, by the rule,

$$27 + 8x^9 = (3 + 2x^3)(9 - 6x^3 + 4x^6).$$

The second factor may be found by dividing $27 + 8x^9$ by the first factor.

Factor:

| | | |
|---|---|---|
| 1. $a^3 + 8$. | 2. $x^3 + 27$. | 3. $64n^3 + 125x^3$. |
| 4. $x^9 + 1$. | 5. $64 + a^6$. | 6. $27a^6 + 512c^3$. |
| 7. $c^6 + x^9$. | 8. $x^6 + m^3$. | 9. $8x^3y^3 + 729z^3$. |
| 10. $8 + y^6$. | 11. $1 + c^3b^6$. | 12. $343 + 216a^3$. |

**96. Factoring a trinomial that is a perfect square.**

Since $(a+b)(a+b) = a^2 + 2ab + b^2$,

and $(a-b)(a-b) = a^2 - 2ab + b^2$,

the square root of $a^2 + 2ab + b^2$ is $a + b$, and

the square root of $a^2 - 2ab + b^2$ is $a - b$.

Now to find these square roots we may proceed thus:

1. *To find the first term of the root, take the square root of the first term of the trinomial.*

2. *To find the second term of the root, divide the middle term of the trinomial by twice the first term of the root.*

CHECK. The square of the second term of the root must be equal to the third term of the trinomial, otherwise the trinomial is not a perfect square.

NOTE. This method of factoring trinomials of this character prepares the student for the process of "completing the square," and extracting square roots of polynomials in general.

Factor $9x^2 - 12xy^2 + 4y^4$.

The square root of $9x^2$ is $3x$, and $-12xy^2 \div 2 \times 3x = -2y^2$; and since the square of $-2y^2$ is $4y^4$, we have

$$9x^2 - 12xy^2 + 4y^4 = (3x - 2y^2)(3x - 2y^2).$$

Factor:

1. $x^2 + 10x + 25$.
2. $9x^2 + 30x + 25$.
3. $x^2 - 12x + 36$.
4. $4x^2 - 24x + 36$.
5. $y^2 + 16y + 64$.
6. $n^2x^2 - 6nx + 9$.
7. $a^2 - 14a + 49$.
8. $a^4 + 4a^2c^2 + 4c^4$.
9. $x^2 - 4xy + 4y^2$.
10. $x^4 - 24x^2 + 144$.
11. $4a^2 + 12ab^2 + 9b^4$.
12. $n^4 + 6m^3n^2 + 9m^6$.
13. $9n^4 - 24c^2n^2 + 16c^4$.
14. $a^6 - 12a^3 + 36$.

**97. Factoring a trinomial that is the product of two binomials having a common term.**

Factoring trinomials of this form is done by reversing the rule of § 82. Note that:

$$(x+5)(x+2) = x^2 + 7x + 10,$$
$$(x+5)(x-2) = x^2 + 3x - 10,$$
$$(x-5)(x+2) = x^2 - 3x - 10,$$
$$(x-5)(x-2) = x^2 - 7x + 10.$$

1. The last term of the trinomial, $+10$, or $-10$, is the product of the last terms of the factors.

2. The coefficient of $x$, viz. $+7$, $+3$, $-3$, $-7$, is the algebraic sum of the last terms of the factors.

3. When the product, 10, is $+$, the last terms of the factors have the same sign, the sign of the middle term.

4. When the product, 10, is $-$, the last terms of the factors have different signs, the greater having the same sign as the middle term of the trinomial.

1. Factor $x^2 - 7x + 12$.

Since 12 is + and 7 is −, the last terms of the factors have the same sign, −.

Now the two numbers whose *sum* is 7 and whose *product* is 12, are 4 and 3.

$$\therefore x^2 - 7x + 12 = (x - 3)(x - 4).$$

2. Factor $x^2 - 5x - 24$.

Since 24 is −, the last terms of the factors have opposite signs, the greater being − because 5 is −.

Now the two numbers whose *difference* is 5 and *product* is 24 are 8 and 3.

$$\therefore x^2 - 5x - 24 = (x - 8)(x + 3).$$

Factor:

| | |
|---|---|
| 1. $x^2 + 3x + 2$. | 2. $x^2 - 10x + 24$. |
| 3. $x^2 - 5x + 4$. | 4. $x^2 + 12x - 45$. |
| 5. $x^2 + 6x + 8$. | 6. $x^2 - 16x + 63$. |
| 7. $x^2 - 5x + 6$. | 8. $x^2 - 2x - 120$. |
| 9. $x^2 + 2x - 8$. | 10. $x^2 + 3x - 108$. |
| 11. $x^2 - 3x - 4$. | 12. $x^2 - 5x - 300$. |
| 13. $x^2 + 7x - 8$. | 14. $x^2 - 20x + 75$. |
| 15. $y^2 - y - 20$. | 16. $n^2 + 18n + 72$. |
| 17. $y^2 + y - 42$. | 18. $n^2 - 3n - 180$. |
| 19. $y^2 - y - 12$. | 20. $a^2 - 20a + 91$. |
| 21. $y^2 + y - 72$. | 22. $a^2 + 2a - 143$. |
| 23. $y^2 + 4y + 4$. | 24. $b^2 - 4b - 320$. |
| 25. $y^2 + 6y + 9$. | 26. $b^2 + 5b - 150$. |
| 27. $z^2 + 2z + 1$. | 28. $z^2 - 19z + 78$. |
| 29. $z^2 - z - 90$. | 30. $z^2 - 8z - 240$. |

## MISCELLANEOUS EXAMPLES

Factor:

1. $x^2 - a^2$.
2. $x^2 + 6x + 9$.
3. $x^2 - 9$.
4. $x^2 - 8x + 16$.
5. $m^2 - 16n^2$.
6. $y^2 + 10ay + 25a^2$.
7. $a^2 - n^2x^4$.
8. $z^2 - 2n^2yz + n^4y^2$.

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9. $6a^2 - 9ab$.
10. $x^2 + 10x + 21$.
11. $4x^2y + 8xy^2$.
12. $y^2 - 12y + 35$.
13. $a^3 + 125x^3$.
14. $x^2 - 2x - 63$.
15. $216 - n^3x^3$.
16. $y^2 + y - 72$.

---

17. $a^4 - b^2$.
18. $a^4 - 4a^2y - 12y^2$.
19. $m^6 + n^3$.
20. $x^6 + 9ax^3 + 14a^2$.
21. $16x^4 - 25y^2$.
22. $y^8 + 3by^4 - 40b^2$.
23. $(a+b)^2 - x^2$.
24. $z^{10} - 12cz^5 + 36c^2$.

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25. $(x^3 + x^2)^2 - (x+3)^2$.
26. $(x^3 - x^2)^2 - (x-1)^2$.
27. $(x^2 + x)^2 - (x-1)^2$.
28. $(a^2 - 2a)^2 - (2a+3)^2$.

---

29. $x^3 + 3x^2 - 4x - 12$.
30. $y^3 - ay^2 + 2y - 2a$.
31. $x^3 + x^2y - anx - any$.
32. $3x^3 - 2x^2 + 15x - 10$.

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33. $x^3 - (x+3)^3$.
34. $a^2 - (b - c + d)^2$.
35. $(a+2)^3 + a^3$.
36. $(x^2 + 4x + 4)^2 - (2x)^2$.

### COMMON DIVISORS OR FACTORS

**98.** 1. Name two divisors of 10; of 15; of 35.

2. Name two factors of 21; of 33; of 51.

3. Name a common divisor of 15 and 21. What is the greatest common divisor of 18 and 24?

4. Is $x$ a common factor of $3x^2$ and $2x^3$? Is $x^2$?

5. Is $x^2$ of a higher degree than $x$?

6. Name the factor of highest degree that is common to $x^3y^4z$ and $x^5y^2z^3$. Is that factor an exact divisor of $x^3y^4z$ and $x^5y^2z^3$?

7. Are $6x^3y$ and $8x^2y^3$ exactly divisible by $xy$? by $x^2y$? by $2x^2y$? Of what degree is $xy$? $x^2y$?

**99.** A **common factor** of two or more numbers is an exact divisor of each of them.

**100.** The **highest common factor** of two or more numbers is the number of highest degree and largest numerical coefficient that will exactly divide each of them.

"Highest common factor" is denoted by H. C. F.

Thus, the common factors of $3x^2y^3$ and $9x^2y$ are $3$, $x$, $y$, $x^2$, $3x$, $3y$, $3x^2$, $xy$, $3xy$, $x^2y$, and $3x^2y$ with sign + or −. But $3x^2y$ is the H. C. F. because it has the highest degree and the largest numerical coefficient. Usually the H. C. F. is considered as positive in sign.

When two numbers have no common factor, they are said to be **prime to each other.**

NOTE. In arithmetic the highest common factor is usually called the greatest common divisor; and there is no objection to using the latter name here, since all letters and combinations of letters are at present supposed to represent integers.

**101.** PRINCIPLE. *The highest common factor of two or more numbers is equal to the product of all their common prime factors, numerical and literal.*

**102. Finding the H. C. F. of two or more numbers.**

1. Find the H. C. F. of $30\,a^4b^2c$ and $12\,a^3b^3d$.

$30\,a^4b^2c = 5 \times 2 \times 3\,aaaabbc = (2 \times 3 \times aaabb)(5\,ac)$.

$12\,a^3b^3d = 2 \times 2 \times 3\,aaabbbd = (2 \times 3 \times aaabb)(2\,bd)$.

$\therefore$ the H. C. F. $= 2 \times 3\,aaabb = 6\,a^3b^3$.

2. Find the H. C. F. of $6(x^2 - 3x - 4)$ and $8(x^2 - 1)$.

$6(x^2 - 3x - 4) = 2 \times 3(x - 4)(x + 1) = 2(x + 1) \times 3(x - 4)$.

$8(x^2 - 1) = 2 \times 4(x + 1)(x - 1) = 2(x + 1) \times 4(x - 1)$.

$\therefore$ the H. C. F. $= 2(x + 1)$.

3. Find the H. C. F. of $4x^2 - 12x + 9$ and $4x^2 - 6x$.

$4x^2 - 12x + 9 = (2x - 3)(2x - 3)$.

$4x^2 - 6x = 2x(2x - 3)$.

$\therefore$ the H. C. F. $= 2x - 3$.

4. Find the H. C. F. of $x^3 - 2x^2 - 24x$ and $3x^2 + 3x - 36$.

$$x^3 - 2x^2 - 24x = x(x^2 - 2x - 24).$$
$$= x(x - 6)(x + 4).$$
$$3x^2 + 3x - 36 = 3(x^2 + x - 12).$$
$$= 3(x - 3)(x + 4).$$

$\therefore$ the H. C. F. $= x + 4$.

5. Find the H. C. F. of $x^2 - 2x - 35$, $x^2 - 9x + 14$, and $x^2 - 49$.

$$x^2 - 2x - 35 = (x - 7)(x + 5).$$
$$x^2 - 9x + 14 = (x - 7)(x - 2).$$
$$x^2 - 49 = (x - 7)(x + 7).$$

$\therefore$ the H. C. F. $= x - 7$.

*To find the H. C. F. of two or more numbers, factor each of the given numbers.*

*Find the product of all the common factors, taking each factor the least number of times it occurs in any of the given numbers.*

Find the H. C. F. of:

1. $2a^2b^3$ and $3a^4b^2$.
2. $20x^3y$ and $15x^4y$.
3. $3x^3y^2$ and $6x^2y^3$.
4. $16a^2b$ and $30ab^2$.
5. $4c^4x^5$ and $2c^3x^6$.
6. $x^2-1$ and $x^2-x-2$.
7. $(x+1)^2$ and $x^2-1$.
8. $x^2-4$ and $x^2+x-2$.
9. $x^3+8$ and $x^2-4$.
10. $x^2-9$ and $x^2-x-6$.

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11. $x^2+2x-3$ and $x^2-5x-24$.
12. $x^2-3x-4$ and $x^2+2x-24$.
13. $x^2-4x-5$ and $x^2+2x-35$.
14. $x^2+7x-8$ and $x^2+3x-40$.

---

15. $x^2+3xy-28y^2$ and $x^2-2xy-63y^2$.
16. $x^2-4xy-96y^2$ and $x^2+3xy-40y^2$.
17. $a^4-5a^2b^2+4b^4$ and $a^2+9ab-10b^2$.
18. $a^4-8a^2b^2-9b^4$ and $a^2+6ab-27b^2$.
19. $c^4-6c^2x+9x^2$ and $c^4+c^2x-12x^2$.

---

20. $a^2-c^2$, $a^2+5ac+4c^2$, and $a^2-9ac-10c^2$.
21. $ax-3a$, $x^2-7x+12$, and $ax^2+5ax-24a$.
22. $x^2-8x+15$, $x^2-4x-5$, and $x^2-3x-10$.
23. $x^2+3x-18$, $x^2-4x+3$, and $x^2+2x-15$.

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24. $a^3-3a^2-4a+12$ and $a^3+2a^2-9a-18$.
25. $a^3-b^2-a^2b+ab$ and $a^4-ab^3+b^4-a^3b$.
26. $39(a^6-b^6)$, $65(a^3+b^3)$, $91(a^2-ab+b^2)$.
27. $a^4-c^4$, $a^3+a^2c-ac^2-c^3$, $a^4-2a^2c^2+c^4$.
28. $5a^3+40b^3$, $7a^3+28a^2b+28ab^2$, $3a^4-12a^2b^2$.

### COMMON MULTIPLES

**103.** 1. Name two numbers that are divisible by 6, 8, and 12.

2. Name the least number that is divisible by both 6 and 9.

3. Are $6x^3y$ and $4x^2y^4$ exact divisors of $24x^4y^4$? of $12x^3y^4$? Of what degree is $24x^4y^4$? $12x^2y^4$?

4. Name the number of lowest degree that $6a^4b^2c$ and $15a^3b^5x$ will exactly divide.

5. Name the number of lowest degree and least numerical coefficient that $9x^2y^4z$ and $12x^6y^5z^2$ will exactly divide.

**104.** A **multiple** of a number is any number that the given number will exactly divide.

Thus, $6a^4b^2$ is a multiple of $ab$, $2a^2b$, $3a^3b^2$, etc.

**105.** A **common multiple** of two or more numbers is any number that the given numbers will exactly divide.

Thus, $24a^3b^4$ is a common multiple of $3a^2b$ and $4ab^3$.

**106.** The **lowest common multiple** of two or more numbers is the number of lowest degree and least numerical coefficient that is exactly divisible by each of the given numbers.

Thus, $12a^2b^3$ is the lowest common multiple of $3a^2b$ and $4ab^3$.

"Lowest common multiple" is denoted by L. C. M.

NOTE. In arithmetic the lowest common multiple is usually called the least common multiple. Either name may be used when all letters and combinations of letters stand for integers.

**107.** PRINCIPLE. *The lowest common multiple of two or more numbers is equal to the product of all their different prime factors, taking each factor the greatest number of times it occurs in any of the given numbers.*

**108. Finding the L. C. M. of two or more numbers.**

1. Find the L. C. M. of $12\,a^2bc$, $18\,ab^3$, and $30\,abc^2$.

$$12\,a^2bc = 2 \times 2 \times 3 \times a^2 \times b \times c.$$
$$18\,ab^3 = 2 \times 3 \times 3 \times a \times b^3.$$
$$30\,abc^2 = 2 \times 3 \times 5 \times a \times b \times c^2.$$

The different prime factors are 2, 3, 5, $a$, $b$, $c$.

Now taking each of these factors the greatest number of times it occurs in any of the given numbers, we have

The L. C. M. $= 2 \times 2 \times 3 \times 3 \times 5 \times a^2 \times b^3 \times c^2 = 180\,a^2b^3c^2$.

2. Find the L. C. M. of $2x^2 - 10x - 28$ and $3x^2 + 9x - 210$.

$$2x^2 - 10x - 28 = 2(x^2 - 5x - 14)$$
$$= 2(x+2)(x-7).$$
$$3x^2 + 9x - 210 = 3(x^2 + 3x - 70)$$
$$= 3(x+10)(x-7).$$
$$\therefore \text{ the L. C. M.} = 6(x+2)(x-7)(x+10).$$

*To find the L. C. M. of two or more numbers, factor each of the given numbers.*

*Find the product of all the different prime factors, taking each factor the greatest number of times it occurs in any of the given numbers.*

Note that the L. C. M. of two numbers is equal to their product divided by their H. C. F.

Thus, the L. C. M. of 18 and 24 $= \dfrac{18 \times 24}{6} = 72$.

Find the L. C. M. of:

1. $3\,a^2bc^3$ and $6\,a^3b^2$.
2. $x^2 - 1$ and $x^2 + x$.
3. $4\,a^4b^2c$ and $9\,b^3c^2$.
4. $a^2 - 1$ and $a^2 + a$.
5. $n^2x^2yz^3$ and $nx^2z^5$.
6. $a^2 - 1^4$ and $(a+1)^2$.
7. $a^3b^2cd$ and $b^2c^2d$.
8. $a^3 - 8$ and $a^2 - 4$.
9. $9\,x^4yz^5$ and $6\,x^5y^2$.
10. $x^3 + 8$ and $x^2 + 2x$.

Find the L. C. M. of:

1. $x^2 - 2x - 15$ and $x^2 + 2x - 35$.
2. $y^2 + 3y - 28$ and $y^2 - 3y - 70$.
3. $z^2 - 4z - 12$ and $z^2 + 2z - 48$.
4. $a^2 + 6a - 16$ and $a^2 - 3a - 88$.
5. $b^2 - 6b - 40$ and $b^2 + 8b + 16$.
6. $x^2 + 8x + 16$ and $x^2 + 6x + 8$.
7. $a^2 - ay - 6y^2$ and $2a^2 - 5ay - 3y^2$.

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8. $x^2 + 2xy - 99y^2$ and $2x^2 - 6xy - 108y^2$.
9. $x^2 - 7xy - 60y^2$ and $3x^2 - 9xy - 324y^2$.
10. $x^2 + 8xy + 16y^2$ and $5x^2 - 5xy - 100y^2$.
11. $a^2 - 5ab - 50b^2$ and $4a^2 - 8ab - 320b^2$.
12. $x^2 + 8xy + 15y^2$ and $x^2 - 2xy - 35y^2$.
13. $a^2 - b^2$, $a^2 + 2ab + b^2$, and $a^2 + ab$.

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14. $3x^2 - 48$, $x^2 - 2x - 24$, and $7x^2 + 56x + 112$.
15. $ax^2 - 9a$, $x^2 + 4x - 21$, and $4x^2 + 28x - 120$.
16. $bx^2 - b^3$, $x^2 - 3bx - 4b^2$, and $x^2 - 9bx - 10b^2$.
17. $16x^2 - 1$, $16x^2 - 8x + 1$, and $20x - 5$.
18. $a^2 - 9$, $a^2 - 4a - 21$, and $a^2 + 6a + 9$.

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19. $a^2 - b^2 + ac - bc$ and $ab - ac + b^2 - c^2$.
20. $4a + ax + 16 - x^2$ and $a^2 - x^2 + 4a + 4x$.
21. $(x+1)(x+2)$, $(x+1)(x-5)$, $(x+2)^2$, $3x - 15$.
22. $(x-3)(x+4)$, $(x+4)(x-6)$, $x^3 + 4x^2 + 2x + 8$.
23. $(x+5)(x+3)$, $x^2 + 9x + 18$, $x^3 + 3x^2 + 3x + 9$.

## IV. FRACTIONS

### DEFINITIONS

**109.** A **fraction** is an indicated quotient, the dividend being written over the divisor with a line between them.

Thus, $a \div b$, when written $\frac{a}{b}$, is a fraction.

**110.** The **numerator** is the dividend, and the **denominator** is the divisor. The **terms** of a fraction are the numerator and denominator.

NOTE. In arithmetic a fraction is generally defined and treated as a part of a unit, but for the more general purposes of algebra it is usually and more conveniently regarded as a quotient. Thus, in arithmetic $\frac{3}{5}$ is defined as 3 of the 5 equal parts of a unit or $\frac{3}{5}$ of a unit, but in algebra we consider it to be 1 of the 5 equal parts of 3 units or $\frac{1}{5}$ of 3 units. Having thus defined it, we must deduce the principles and rules of fractions accordingly.

### FUNDAMENTAL PRINCIPLES

**111.** Evidently, $2a$ is contained in $12a$ the same number of times that 2 is contained in 12.

That is, $$\frac{12a}{2a} = \frac{12a \div a}{2a \div a} = \frac{12}{2}.$$

Again, 4 is evidently contained in 12 the same number of times that $4n$ is contained in $12n$,

That is, $$\frac{12}{4} = \frac{12 \times n}{4 \times n} = \frac{12n}{4n}.$$

From the above we infer:

*Dividing or multiplying both terms of a fraction by the same integer does not alter the value of the fraction.*

## REDUCTION

**112. Reducing** a fraction is changing its terms or form without altering its value.

**113. Reducing fractions to their lowest terms.**

A fraction is in its **lowest terms** when the numerator and denominator have no common factor.

**Canceling** a common factor in both terms is equivalent to dividing both terms by that factor. Hence,

*Factor the terms and cancel all common factors.*

Reduce to lowest terms $\frac{9}{12}$; $\frac{x^2-2x-15}{x^2-9}$.

$$\frac{9}{12}=\frac{3\times 3}{3\times 4}=\frac{3}{4};\quad \frac{x^2-2x-15}{x^2-9}=\frac{(x+3)(x-5)}{(x+3)(x-3)}=\frac{x-5}{x-3}.$$

Reduce to lowest terms:

1. $\frac{18}{24}$.
2. $\frac{15\,a^2b}{20\,ab^2}$.
3. $\frac{42\,m^3n^4rx^5}{56\,m^2nx^2y}$.
4. $\frac{105}{231}$.
5. $\frac{24\,x^5y^2}{36\,xy^4}$.
6. $\frac{54\,c^3d^5x^3y}{72\,cd^2x^6y^3}$.
7. $\frac{6\,a^2b-8\,ab^2}{2\,a^3b-4\,a^2b}$.
8. $\frac{x^2-2x-120}{x^2-3x-108}$.
9. $\frac{9\,x^3y+6\,x^2y^2}{6\,x^2y-3\,x^4y}$.
10. $\frac{a^2+3a-180}{a^2+5a-150}$.
11. $\frac{9\,x^2-4\,y^2}{9\,x^2+6\,xy}$.
12. $\frac{(x+y)^2-z^2}{ax+ay+az}$.
13. $\frac{4\,x^2-8\,x}{x^2+x-6}$.
14. $\frac{a^2-(b-c)^2}{na+nb-nc}$.
15. $\frac{a^2+5a}{a^2-a-30}$.
16. $\frac{x^2+10x+21}{x^2+2x-35}$.

**114. Reducing fractions to integers or mixed numbers.**

1. Reduce $\frac{28}{7}$ and $\frac{31}{8}$ to integers or mixed numbers.

$\frac{28}{7} = 28 \div 7 = 4$, an integer; $\frac{31}{8} = 31 \div 8 = 3\frac{7}{8}$, a mixed number.

2. Reduce $\dfrac{x^2-9}{x-3}$ and $\dfrac{x^2+6x+7}{x+2}$ to integers or mixed numbers.

Performing the indicated divisions, we obtain,

(*a*) $$\frac{x^2-9}{x-3} = x+3, \text{ an integer.}$$

(*b*) $$\frac{x^2+6x+7}{x+2} = x+4-\frac{1}{x+2}, \text{ a mixed number.}$$

*Divide the numerator by the denominator.*

NOTE. If there is a remainder, its division by the denominator must be indicated, and the fraction, thus formed, must be annexed, with its proper sign, to the integral part of the quotient.

Reduce to integers or mixed numbers:

1. $\frac{15}{3}$.
2. $\dfrac{x^2+3x}{x}$.
3. $\dfrac{x^2-1}{x+1}$.
4. $\frac{45}{7}$.
5. $\dfrac{x^2+2x}{x+1}$.
6. $\dfrac{x^3+8}{x+2}$.
7. $\dfrac{x^2+11x+30}{x+6}$.
8. $\dfrac{x^3-6x^2+11x-6}{x^2-3x+2}$.
9. $\dfrac{x^2-3x-180}{x+12}$.
10. $\dfrac{a^4+a^2b^2+b^4}{a^2-ab+b^2}$.
11. $\dfrac{x^2-4x-100}{x-11}$.
12. $\dfrac{x^3+x^2+7x+7}{x^2+7}$.
13. $\dfrac{2x^2+3x+1}{2x+1}$.
14. $\dfrac{3x^2+5x+8}{x^2+2}$.
15. $\dfrac{12x^2}{x+3}$.
16. $\dfrac{8a^3+125}{2a+5}$.

**115. Reducing integers or mixed numbers to fractions.** An integer or a mixed number is reduced to a fraction by reversing the preceding rule.

Thus, since $\frac{4}{3} = 4$, to reduce 4 to thirds we multiply 4 by 3, and write the product over 3, $\frac{12}{3}$.

Again, since $\frac{13}{5} = 2\frac{3}{5}$, to reduce $2\frac{3}{5}$ to fifths we multiply 2 by 5, add 3 to the product, and write the sum over 5.

1. Reduce $a + \frac{b}{c}$ to a fractional form.

$a$ multiplied by $c$ plus $b = ac + b$; $\frac{ac+b}{c}$. *Ans.*

2. Reduce $2x + 3 - \frac{x-5}{x}$ to a fractional form.

$x(2x+3) - (x-5) = 2x^2 + 3x - x + 5 = 2x^2 + 2x + 5$;

$\frac{2x^2+2x+5}{x}$. *Ans.*

Note that $-$ before the fraction changes the signs in the numerator.

*Multiply the entire part by the denominator of the fraction; add to this product the numerator if the sign of the fraction is $+$, or subtract from it the numerator if the sign is $-$; write the result over the denominator.*

Reduce to fractional forms:

1. $6\frac{3}{4}$.
2. $5x + \frac{x}{2}$.
3. $3x + \frac{2x-5}{4}$.
4. $7 - \frac{3}{b}$.
5. $6x - \frac{5x}{7}$.
6. $5x - \frac{3x+4}{7}$.
7. $a + \frac{b}{x}$.
8. $4a + \frac{cx}{d}$.
9. $7c - \frac{3a-b}{x}$.
10. $3x + \frac{2x-3}{x+1}$.
11. $a + x + \frac{a^2+x^2}{a-x}$.
12. $5a - \frac{4a+7}{2a+3}$.
13. $a + c - \frac{2ac+c^2}{a+c}$.

**116. Reducing dissimilar to similar fractions.**

**Similar fractions** are fractions that have the same denominator.

**Dissimilar fractions** are fractions that have not the same denominator.

1. Reduce $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{7}{8}$ to similar fractions.

Since multiplying both terms of a fraction by the same number does not alter its value, we multiply the terms by such numbers as will make the denominators the same. Thus, by multiplying both terms of $\frac{3}{4}$ by 6, both terms of $\frac{5}{6}$ by 4, and both terms of $\frac{7}{8}$ by 3, the fractions are reduced to the same denominator, 24.

$$\frac{3}{4}=\frac{3\times 6}{4\times 6}=\frac{18}{24}$$
$$\frac{5}{6}=\frac{5\times 4}{6\times 4}=\frac{20}{24}$$
$$\frac{7}{8}=\frac{7\times 3}{8\times 3}=\frac{21}{24}$$

Note that 24 is the lowest common multiple of the denominators 4, 6, 8; and the multipliers 6, 4, 3 are the quotients of 24 by the denominators, respectively. Thus, $6 = 24 \div 4$, $4 = 24 \div 6$, $3 = 24 \div 8$.

2. Reduce $\frac{3}{1+2x}$, $\frac{2x}{1-2x}$, and $\frac{3-x}{1-4x^2}$ to similar fractions.

The L. C. M. of the denominators is $1-4x^2$, which is to be the common denominator.

The numerators are found thus:

(*a*) $1-4x^2 \div (1+2x) = 1-2x$; $3\times(1-2x) = 3-6x$, 1st numerator.

(*b*) $1-4x^2 \div (1-2x) = 1+2x$; $2x\times(1+2x) = 2x+4x^2$, 2d numerator.

(*c*) $1-4x^2 \div (1-4x^2) = 1$; $1\times(3-x) = 3-x$, 3d numerator.

Therefore the required fractions are

$$\frac{3-6x}{1-4x^2},\quad \frac{2x+4x^2}{1-4x^2},\quad \frac{3-x}{1-4x^2}.$$

*Find the lowest common multiple of the denominators of the fractions for the lowest common denominator. Divide this denominator by the denominator of each fraction, and multiply the terms of the fraction by the quotient.*

Reduce to similar fractions having their lowest common denominator:

1. $\frac{5}{8}$ and $\frac{7}{12}$.
2. $\frac{m}{a^2b}$ and $\frac{n}{ab^2}$.
3. $\frac{3a}{4}$ and $\frac{5x}{6}$.
4. $\frac{5}{ab}$, $\frac{6}{ac}$, $\frac{7}{bc}$.
5. $\frac{c}{ab}$ and $\frac{b}{ac}$.
6. $\frac{ax}{y}$, $\frac{bc}{ax}$, $\frac{bd}{ay}$.
7. $\frac{2a}{9x}$ and $\frac{5b}{12y}$.
8. $\frac{a+b}{a-b}$, $\frac{a-b}{a+b}$.
9. $\frac{x+y}{3x}$, $\frac{3x}{x-y}$.
10. $\frac{4}{x+2}$, $\frac{3}{x-2}$, $\frac{1}{x}$.
11. $\frac{3}{x+1}$, $\frac{5}{x-1}$.
12. $\frac{2x}{x^3-1}$, $\frac{5}{x^2+x+1}$.
13. $\frac{x+2}{x^2-9}$, $\frac{3}{2x+6}$, $\frac{5}{2x-6}$.
14. $\frac{x+1}{x^2-x-6}$, $\frac{x-1}{x^2+x-12}$, $\frac{2}{x+4}$.
15. $\frac{x}{x^2+3x+2}$, $\frac{x}{x^2+5x+6}$, $\frac{3}{x+3}$.
16. $\frac{x}{3x+6}$, $\frac{3x}{4x-8}$, $\frac{2}{12x^2-48}$.
17. $\frac{3}{x^4-y^4}$, $\frac{2}{x^2-y^2}$, $\frac{1}{x+y}$.
18. $\frac{a}{a^3+b^3}$, $\frac{a-b}{a^2-ab+b^2}$, $\frac{1}{a^2+ab}$.
19. $\frac{x-6}{x^2+6x+8}$, $\frac{x-4}{x^2+8x+12}$, $\frac{x-2}{x^2+10x+24}$.

## ADDITION AND SUBTRACTION

**117.** 1. Find the sum and difference of $\frac{c}{n}$ and $\frac{d}{n}$.

According to the distributive law in division (§ 37), we have

(*a*) $$\frac{c}{n}+\frac{d}{n}=(c+d)\div n=\frac{c+d}{n},$$

(*b*) $$\frac{c}{n}-\frac{d}{n}=(c-d)\div n=\frac{c-d}{n}.$$

*To add or subtract two similar fractions, add or subtract, respectively, their numerators, and write the sum or difference over their common denominator.*

2. Find the sum and difference of $\frac{3x}{4a}$ and $\frac{2y}{3b}$.

$\frac{3x}{4a}=\frac{9bx}{12ab}$
$\frac{2y}{3b}=\frac{8ay}{12ab}$

Reduce the fractions to a common denominator. Then, the fractions being similar, we have

(*a*) $$\frac{3x}{4a}+\frac{2y}{3b}=\frac{9bx}{12ab}+\frac{8ay}{12ab}=\frac{9bx+8ay}{12ab}.$$

(*b*) $$\frac{3x}{4a}-\frac{2y}{4a}=\frac{9bx}{12ab}-\frac{8ay}{12ab}=\frac{9bx-8ay}{12ab}.$$

*To add or subtract dissimilar fractions, reduce the fractions to similar fractions, and then proceed according to the previous rule.*

**1.** $\frac{3}{4}+\frac{7}{10}$.

**2.** $\frac{x}{3}+\frac{x}{3}$.

**3.** $\frac{x-3}{4}+\frac{x+2}{3}$.

**4.** $\frac{2x}{5}-\frac{x}{8}$.

**5.** $\frac{3y}{8}-\frac{2y}{7}$.

**6.** $\frac{x+8}{3}-\frac{x+3}{5}$.

**7.** $\frac{3a}{7}-\frac{2a}{11}$.

**8.** $\frac{2b}{15}+\frac{4b}{9}$.

**9.** $\frac{a-6}{8}-\frac{a-3}{12}$.

**10.** $\frac{3}{2a}+\frac{4}{3a}$.

**11.** $\frac{5}{6x}-\frac{3}{8x}$.

**12.** $\frac{a+3}{12x}+\frac{a-2}{8x}$.

Find the following:

1. $\frac{x+4}{3}+\frac{x-5}{4}$.

2. $\frac{x-5}{4}-\frac{x+6}{5}+\frac{x-3}{8}$.

3. $\frac{2x+1}{3}-\frac{x-5}{6}$.

4. $\frac{x+3y}{4x}+\frac{x-5y}{6x}$.

5. $\frac{3x-y}{10}+\frac{x+2y}{3}-\frac{8x+3y}{15}$.

6. $\frac{a+2b}{5b}+\frac{3a-b}{6b}-\frac{2a+3b}{10b}$.

7. $\frac{a^2-bc}{bc}+\frac{b^2-ac}{ac}-\frac{c^2-2bc}{bc}$.

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8. $\frac{1}{x+1}-\frac{2}{x+2}+\frac{3}{x+3}$.

9. $\frac{2}{x+5}-\frac{3}{x+4}+\frac{4}{x+3}$.

10. $\frac{8}{9-x^2}-\frac{2}{3-x}+\frac{2}{3+x}$.

11. $\frac{x+3}{x-3}+\frac{x-3}{x+3}-\frac{x^2+9}{x^2-9}$.

12. $\frac{x+3}{x^2+5x+6}-\frac{x+4}{x^2+8x+15}$.

13. $\frac{x+2}{x^2-1}+\frac{x}{x^2-x-2}-\frac{2}{x-2}$.

14. $\frac{25}{x^2-7x+12}+\frac{3}{x^2-9}-\frac{28}{x^2-16}$.

15. $\frac{m-n}{mn}+\frac{n-p}{np}+\frac{p-m}{mp}$.

16. $\frac{x^2+x}{x^3-1}-\frac{1}{x-1}+\frac{1}{x^2+x+1}$.

## MULTIPLICATION AND DIVISION

**118.** 1. How much is $\frac{a}{b}+\frac{a}{b}+\frac{a}{b}$? $3\times\frac{a}{b}$? $n\times\frac{a}{b}$?

2. Find, in lowest terms, $3\times\frac{a}{6b}$; $n\times\frac{a}{nb}$. In what two ways, then, may a fraction be multiplied by an integer?

*Multiplying the numerator or dividing the denominator of a fraction by an integer, multiplies the fraction by the integer.*

**119.** Since $n$ times $\frac{a}{b}=\frac{na}{b}$, then $\frac{na}{b}\div n=\frac{a}{b}$,

and since $n$ times $\frac{a}{bn}=\frac{a}{b}$, then $\frac{a}{b}\div n=\frac{a}{bn}$. In what two ways, then, may a fraction be divided by an integer?

*Dividing the numerator or multiplying the denominator of a fraction by an integer, divides the fraction by the integer.*

An integer may be written as a fraction by writing 1 under it for a denominator. Thus, $a=\frac{a}{1}$.

**120.** One fraction is the **reciprocal** of another when the numerator of either is the denominator of the other.

Thus, the reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$; of 8 or $\frac{8}{1}$, $\frac{1}{8}$; of $2\frac{1}{2}$ or $\frac{5}{2}$, $\frac{2}{5}$.

**121.** To **invert** a fraction is to take its reciprocal.

To multiply a number by $\frac{1}{n}$ is to take $\frac{1}{n}$ of the number or divide the number by $n$.

Thus, $12a\times\frac{1}{3}$ is $\frac{1}{3}$ of $12a$, or $12a\div 3$, which is $4a$.

Again, $\frac{a}{b}\times\frac{1}{n}$ is $\frac{1}{n}$ of $\frac{a}{b}$, or $\frac{a}{b}\div n$, which is $\frac{a}{bn}$.

*To divide a fraction by an integer, multiply the fraction by the reciprocal of the integer.*

Find the following:

1. $7\frac{1}{2} + 1\frac{1}{4}$.

2. $\left(a - \frac{1}{a}\right) \div \left(1 + \frac{1}{a}\right)$.

3. $5\frac{3}{5} \times 2\frac{1}{2}$.

4. $\left(2 - \frac{8}{a^2}\right) \times \frac{3}{a-2}$.

5. $\frac{2}{3} \times \frac{9}{10} \times \frac{2\,5}{2\,7}$.

6. $\frac{a}{bc} \times \frac{b}{ac} \times \frac{c}{ab}$.

7. $\frac{x^2 + 8x}{x^2 - 16} \div \frac{x}{x+4}$.

8. $\frac{(x+2)^2}{x^2 - 49} \div \frac{x+2}{x+7}$.

9. $\frac{x^3 + 8}{x^3 - 8} \div \frac{x+2}{x-2}$.

10. $\frac{a^3 - 1}{a^3 + 1} \times \frac{a^2 - ax + 1}{a^2 + ax + 1}$.

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11. $\frac{x^2 - 4}{x^2 - 9} \times \frac{x^2 - 2x - 15}{x^2 - 3x - 10}$.

12. $\frac{x^2 - 25}{x^2 - 64} \div \frac{x^2 + 11x + 30}{x^2 + 2x - 24}$.

13. $\frac{x^2 + 3x + 2}{x^2 + 5x + 6} \times \frac{x^2 + 7x + 12}{x^2 + 9x + 20}$.

14. $\frac{a^2 + 7a + 12}{a^2 + 9a + 20} \div \frac{a^2 - 5a - 24}{a^2 - 3a - 40}$.

15. $\frac{a^3 - 64}{a^3 + 64} \times \frac{a^2 - 4a + 16}{a^2 + 4a + 16} \div \frac{a-4}{a+4}$.

Invert divisor and multiply the three fractions.

16. $\frac{x^4 - y^4}{a^4 - b^4} \times \frac{a^2 + b^2}{x^2 + y^2} \div \frac{x-y}{a+b}$.

17. $\frac{a^4 - x^4}{a^2 - 2ax + x^2} \div \frac{a^2x + x^3}{a^3 - x^3}$.

18. $\frac{x^2 - x - 12}{x^2 - 9} \times \frac{x^2 - 2x - 3}{x^2 - 2x - 8} \div \frac{x^2 + x}{x+2}$.

19. $\frac{c^2 - cx - c + x}{2 - 3c + c^2} \times \frac{c^3 - 8}{c^3 - x^3} \div \frac{8 - c^3}{c^2 + cx + x^2}$.

**124. Reducing complex to simple fractions.**

A **complex fraction** is a fraction that has a fraction in one or both of its terms.

Thus, $\dfrac{\frac{2}{3}}{\frac{3}{4}}$, $\dfrac{\frac{4}{5}}{\frac{7}{8}}$, $\dfrac{a}{\frac{b}{c}}$, $\dfrac{\frac{a+1}{b}}{c+d}$, and $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ are complex fractions.

1. Simplify $\dfrac{3\frac{1}{3}+4\frac{1}{2}}{5\frac{3}{4}}$.

This fraction means $(3\frac{1}{3}+4\frac{1}{2}) \div 5\frac{3}{4}$.

In general, to simplify a complex fraction is to solve a problem in division of fractions.

The shortest way to reduce fractions like the above is to multiply both terms (§ 111) by the L. C. M. of the denominators of the fractions in both terms.

Thus, multiplying both terms of the above fraction by 12, the L. C. M. of 3, 2, and 4, we have

$$\frac{40+54}{69}, \text{ which is equal to } \frac{94}{69} \text{ or } 1\tfrac{25}{69}.$$

2. Simplify $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}$.

The L. C. M. of the denominators $a-b$ and $a+b$ is $(a-b)(a+b)$.

Multiplying both terms of the given fraction by $(a-b)(a+b)$ gives

$$\frac{(a^2+2ab+b^2)-(a^2-2ab+b^2)}{(a^2+2ab+b^2)+(a^2-2ab+b^2)}$$

$$=\frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2+2ab+b^2+a^2-2ab+b^2}$$

$$=\frac{4ab}{2a^2+2b^2}$$

$$=\frac{2ab}{a^2+b^2}.$$

Reduce to simple fractions:

1. $\dfrac{\frac{x}{2}+\frac{y}{3}}{\frac{x}{3}-\frac{y}{4}}$.

2. $\dfrac{\frac{x}{a}+\frac{y}{b}}{\frac{x}{c}-\frac{y}{d}}$.

3. $\dfrac{3x+\frac{x}{2}}{4y-\frac{y}{3}}$.

4. $\dfrac{a+\frac{3b}{4}}{a-\frac{2b}{3}}$.

5. $\dfrac{\frac{2}{a}+\frac{2}{b}}{\frac{2}{a}-\frac{2}{b}}$.

6. $\dfrac{\frac{2}{ab}+\frac{3}{ac}}{\frac{2}{ab}-\frac{3}{ac}}$.

7. $\dfrac{1+\frac{1}{x+1}}{1-\frac{1}{x-1}}$.

8. $\dfrac{a+\frac{a}{a+1}}{a-\frac{a}{a-1}}$.

9. $\dfrac{\frac{a}{b}+\frac{a+b}{a-b}}{\frac{a}{b}-\frac{a-b}{a+b}}$.

10. $\dfrac{x-1-\frac{4}{x-4}}{x+2+\frac{6}{x-3}}$.

11. $\dfrac{1-\frac{3}{x}-\frac{40}{x^2}}{1+\frac{1}{x}-\frac{72}{x^2}}$.

12. $\dfrac{\frac{x+1}{x-1}+\frac{x-1}{x+1}}{\frac{x+1}{x-1}-\frac{x-1}{x+1}}$.

13. $\dfrac{\frac{x}{8}-\frac{8}{x}}{\frac{x}{8}-\frac{7}{4}+\frac{6}{x}}$.

14. $\dfrac{\frac{x+2}{y+2}-\frac{y+2}{x+2}}{\frac{1}{y+2}-\frac{1}{x+2}}$.

15. $\dfrac{\frac{x}{3}+\frac{2x+5}{x+2}}{\frac{x}{3}+\frac{3x+8}{x+2}}$.

16. $\dfrac{c^2-c+\frac{c-1}{c+1}}{c+\frac{1}{c+1}}$.

## FRACTION PROBLEMS

1. Find the sum of $\frac{1}{2}$ of $a$ dollars and $\frac{1}{3}$ of ($a$ minus 6) dollars.

2. A and B together own a lot of land. If B owns $\frac{3}{n}$ of it, what part of it does A own?

3. Henry paid $\frac{3}{4}$ of a dollar for a knife and $\frac{n}{5}$ of a dollar for a ball. How much more did the knife cost than the ball?

4. Find the cost of $6x$ hats, at $\frac{2n}{3x}$ dollars each.

5. A man having $n$ dollars paid out $\frac{1}{3}$ of it, and then $\frac{1}{5}$ of the remainder. How much had he left?

6. A table costs $a$ dollars, a desk $b$ dollars, and a chair $\frac{1}{3}$ of the difference between the costs of the table and the desk. What was the cost of all?

7. If $\frac{5}{n}$ of a number is $30a$, what is the number?

8. A man walked first $\frac{1}{2}$ of $a+b$ miles and then $\frac{1}{3}$ of $a-b$ miles. How far did he walk all together?

9. How far is it from Lisbon to Trenton, if $\frac{a}{2}+\frac{b}{3}$ miles is $\frac{1}{6}$ of the distance?

10. What is the cost of a coat, vest, and trousers, if the coat costs $a$ dollars, the vest $(a-10)$ dollars, and the trousers $\frac{3}{4}$ as much as the coat and vest together?

11. Express $(bc-ac+ab) \div abc$ as the sum of three fractions in their lowest terms.

12. Express $\frac{x}{2}-\frac{2}{3}+\frac{5}{4x}$ as a trinomial divided by a monomial.

**13.** A boy has $x$ cents. If he spends 6¢, and then gives his sister $\frac{1}{3}$ of what he has left, how much will he have?

**14.** There are $x$ trees in an orchard, of which $\frac{1}{3}$ are apple trees, $\frac{1}{4}$ of the remainder pear trees, and all the rest peach trees. How many peach trees are there?

**15.** Ann has $x$ roses and Mary has $x + 5$ roses. If Mary gives Ann $\frac{1}{3}$ of her roses, how many more roses will Ann then have than Mary?

**16.** Take the fraction $\frac{2}{5}$, and form a new fraction of it by adding $x$ to both of its terms. How much greater is the new fraction than the old one?

**17.** Take the fraction $\frac{x}{x+5}$, and form a new fraction of it by subtracting 3 from both its terms. How much less is the new fraction than the old one?

**18.** A has $x$ dollars and B has $y$ dollars. If A gives B $\frac{1}{2}$ of his money, and B in turn gives A $\frac{1}{3}$ of his money, how much will each have?

**19.** A and B together have $50, of which A has $x$ dollars. If A gives $\frac{1}{2}$ of what he has for a clock, and B gives $\frac{1}{3}$ of what he has for a desk, how much more will the clock cost than the desk?

**20.** A man paid $4 each for $x$ sheep. After 20 had died he sold $\frac{1}{3}$ of the number left at $3 each, and the rest at $7 each. How much did he gain on the whole?

**21.** A father is now $x$ years old, and his son's age is $\frac{3}{5}$ of his father's. What part of the father's age was the son's age 15 years ago?

**22.** If $m$ hats cost $n$ dollars, how much will $x$ hats cost at the same rate?

**23.** A man paid $(n^2 - 1)$ dollars for $n + 1$ pigs. What would be the cost, at the same rate, of $n + 2$ pigs?

**24.** A man works $n$ weeks and a boy 6 weeks. If the man's wages are $d$ dollars per week and the boy's $\frac{1}{3}$ as much, how much do both receive for the work?

**25.** A farmer has 60 sheep. If he sells $n$ of them at $\$5\frac{1}{2}$ each, and the remainder at $\$6\frac{3}{4}$ each, how much will he receive for all of them?

**26.** A lady bought a piano for $n$ dollars, paying $\frac{9}{n}$ dollars cash, and agreeing to pay the remainder in $n - 3$ equal monthly payments. Find the monthly payment.

**27.** If the circumference of a wheel is $1 + \frac{4}{x}$ feet, how many times will the wheel revolve in rolling a distance of $x - \frac{16}{x}$ feet?

**28.** A merchant sold $n$ caps for $(n^2 - 5n)$ dollars. How many caps should he sell for $(n^2 - 25)$ dollars?

**29.** A man had a tract of land containing $x - 4$ acres. After reserving $\frac{21}{x}$ acres for himself, he divided the remainder equally among his $x + 3$ children. Find the number of acres each child received.

NOTE. The minute hand of a clock moves 12 times as fast as the hour hand.

**30.** When it is $n$ minutes after 3 o'clock, $n$ being less than 15, how many minute spaces is the hour hand ahead of the minute hand?

**31.** When it is $x$ minutes after 6 o'clock, $x$ being more than 35, how many minute spaces is the minute hand ahead of the hour hand?

**32.** A can do a piece of work in 4 days and B can do it in $x$ days.

($a$) What part of the work can both together do in 1 day? ($b$) How long will it take both to do the work?

($a$) In 1 day A can do $\frac{1}{4}$ of the work,

and in 1 day B can do $\frac{1}{x}$ of the work.

Then in one day A and B together can do $\frac{1}{4}+\frac{1}{x}$ or $\frac{x+4}{4x}$ of the work, *Ans.*

($b$) If the work were divided into $4x$ equal parts, A and B together could do $x+4$ of the parts in one day; hence they can do the entire work in as many days as $x+4$ is contained times in $4x$; that is, $\frac{4x}{x+4}$ days, *Ans.*

**33.** Find the answers to problem 32 when $x=12$.

HINT. Substitute 12 for $x$ in the preceding answers.

**34.** A can do a piece of work in $x$ days and B can do it in $n$ days. ($a$) What part of the work can A and B together do in 1 day? in 2 days? in $c$ days? ($b$) In how many days can they together do the work?

**35.** Find the answers to problem 34 when $x=8$, $n=24$, and $c=4$.

**36.** A, B, C can do a piece of work in $4x$, $6x$, $8x$ days, respectively. ($a$) What part of the work can they together do in 3 days? ($b$) In how many days can they together complete the work?

**37.** A and B can do a piece of work in $6x$ and $8x$ days, respectively. What part of the work will be done if A works at it 3 days and B 2 days?

**38.** A cistern can be filled by two pipes together in $x$ hours, and by one of the pipes in 10 hours. What part of the cistern will the other pipe fill in 1 hour?

## V. FRACTIONAL EQUATIONS

### 125. Solving fractional equations.

1. Solve $\frac{x}{6}+\frac{x+4}{4}=\frac{x}{3}+2.$

Multiply by 12, the L. C. M. of the denominators.

The result is $2x+3x+12=4x+24.$

Transpose, $2x+3x-4x=24-12.$

Reduce, $x=12.$

An equation is cleared of fractions by multiplying both members by the L. C. M. of the denominators.

2. Solve $\frac{2x+1}{4}-\frac{4x-1}{10}+1\frac{1}{4}=0.$

Multiply by 20, $5(2x+1)-2(4x-1)+25=0.$

Reduce, $10x+5-8x+2+25=0.$

Transpose, $10x-8x=-5-2-25.$

Combine, $2x=-32.$

Divide by 2, $x=-16.$

Note that in clearing the equation of fractions the $-$ sign before the second fraction changes the sign of $4x$ from $+$ to $-$, and the sign of $-1$ from $-$ to $+$ (§ 28).

NOTES. 1. Instead of encouraging the pupils to say "clearing the equation of fractions," let them say "multiplying both members by the L. C. M. of the denominators," for this designates the operation they are performing and calls for the axiom on which it is based.

2. Having treated of fractions, we may hereafter consider that letters may stand for either integers or fractions.

Solve and check:

1. $2x+\frac{x}{3}=14.$
2. $2x-11=16+\frac{x}{5}.$
3. $4x-\frac{x}{2}=21.$
4. $\frac{1}{2}x+\frac{4}{5}x=57-\frac{x}{8}.$
5. $\frac{2x}{5}+\frac{x}{6}=34.$
6. $\frac{2}{3}x+5x=24-\frac{x}{3}.$
7. $\frac{3x}{4}-\frac{x}{5}=33.$
8. $53-\frac{x}{6}=3x-\frac{2}{9}x.$

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9. $\frac{5x}{8}+\frac{x}{3}=46.$
10. $\frac{2}{3}x+\frac{3}{4}x-15=\frac{x}{6}.$
11. $\frac{5x}{6}=\frac{x}{9}+13.$
12. $\frac{3}{4}x-\frac{x}{2}=14-\frac{5}{8}x.$
13. $\frac{3x}{10}=1\frac{1}{2}-\frac{x}{5}.$
14. $\frac{x+7}{3}=3+\frac{x-3}{2}.$
15. $\frac{x}{4}=\frac{55-2x}{3}.$
16. $\frac{5}{2}x-\frac{3}{4}x=\frac{x}{12}+5.$

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17. $\frac{x}{5}=\frac{46-3x}{8}.$
18. $\frac{x+5}{3}-\frac{9-x}{4}=3\frac{1}{2}.$
19. $3x-\frac{2x-5}{3}=11.$
20. $\frac{x+3}{2}-\frac{7x-5}{5}=\frac{2x+3}{10}.$
21. $14+\frac{5-x}{4}=15x.$
22. $\frac{x-4}{4}-\frac{x-1}{3}=\frac{x-26}{5}.$
23. $12-\frac{6+x}{3}=\frac{2}{3}x.$
24. $3x-\frac{x+8}{5}=2-\frac{x-8}{3}.$
25. $\frac{3x}{4}-\frac{3x-2}{6}=\frac{4}{3}x.$
26. $\frac{x-1}{2}+\frac{x-3}{4}-\frac{x-2}{3}=\frac{2}{3}.$

Solve :

1. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=14.$

2. $2x-\frac{x-2}{10}=x+\frac{x+18}{15}.$

3. $\frac{3x}{4}+\frac{2x}{3}-\frac{x}{6}=\frac{5}{2}.$

4. $\frac{3}{4}-\frac{x-2}{3}=\frac{5}{4}-\frac{x+3}{4}.$

5. $\frac{3x-1}{4}=\frac{2x+1}{3}.$

6. $\frac{60-x}{14}-\frac{3x-5}{7}=\frac{3x}{4}.$

7. $\frac{5}{2x+1}=\frac{7}{3x+1}.$

8. $\frac{3}{x}+\frac{5}{2x}+\frac{8}{3x}=1+\frac{19}{6x}.$

9. $\frac{1}{x-1}-\frac{1}{x+1}=\frac{x}{x^2-1}.$

10. $\frac{1}{x-2}+\frac{3}{x+2}=\frac{12}{x^2-4}.$

11. $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}.$

12. $\frac{3}{x+5}+\frac{5}{x-4}=\frac{53}{x^2+x-20}.$

13. $\frac{7x+1}{x+3}-\frac{2x-3}{3(x+3)}=\frac{44}{15}.$

14. $\frac{1}{2x+1}=\frac{13}{4x^2-1}+\frac{2}{1-2x}.$

15. $\frac{1}{6}(x+2)-\frac{2}{3}\left(\frac{2x+1}{2}\right)=\frac{3x-1}{3}.$

16. $\frac{1}{8}(x+5)+\frac{1}{2}(2x-1)=\frac{1}{2}(x+\frac{1}{2}).$

17. $\frac{1}{5}(x-1)-\frac{1}{6}(43-5x)=\frac{1}{8}(3x-1).$

18. $\frac{1}{4}(3x+1)-\frac{1}{7}(5x-4)=\frac{1}{3}(38-7x).$

19. $x-\frac{1}{8}(x-2)=\frac{1}{4}(x+23)-\frac{1}{5}(10+x).$

### PROBLEMS

**1.** The sum of $\frac{1}{4}$ and $\frac{1}{5}$ of a certain number is 45. Find the number.

Let $x =$ the number.

Then $\frac{x}{4} = \frac{1}{4}$ of the number,

and $\frac{x}{5} = \frac{1}{5}$ of the number.

Therefore, $\frac{x}{4} + \frac{x}{5} = 45.$

Multiply by 20, $5x + 4x = 900.$

Whence, $x = 100$, *Ans.*

*Check.* $\frac{100}{4} + \frac{100}{5} = 25 + 20 = 45.$

**2.** The sum of $\frac{1}{3}$ and $\frac{1}{12}$ of a certain number is 65. Find the number.

**3.** John gave $\frac{1}{6}$ of his money for a hat and $\frac{1}{8}$ of it for a pair of shoes. The hat and shoes together cost \$7. How much had he at first?

**4.** The difference between $\frac{1}{6}$ and $\frac{1}{10}$ of a certain number is 7. Find the number.

**5.** Find the length of a rope such that 10 ft. more than $\frac{1}{9}$ of its length is equal to $\frac{1}{8}$ of its length.

**6.** Ben's age is $\frac{1}{4}$ of his mother's age, and the sum of their ages is 45 yr. Find their ages.

**7.** If to $\frac{1}{3}$ of a number you add $\frac{1}{4}$ of it, the sum will be 5 more than $\frac{3}{8}$ of the number. Find the number.

**8.** After spending $\frac{1}{3}$ and $\frac{1}{5}$ of my money I had \$14 left. How much had I at first?

HINT. What I spent + what I had left = what I had at first.

**9.** After spending $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{8}$ of my money I had \$1.40 left. How much had I at first?

**10.** Divide 42 into two parts so that $\frac{2}{3}$ of the first part shall be equal to $\frac{1}{2}$ of the second part.

Let $x =$ first part,

then $42 - x =$ second part.

$\frac{2x}{3} = \frac{2}{3}$ of first part,

and $\frac{42 - x}{2} = \frac{1}{2}$ of second part.

$\therefore \frac{2x}{3} = \frac{42 - x}{2}$.

Multiply by 6, $4x = 126 - 3x$.

Whence, $x = 18$, first part,

$42 - 18 = 24$, second part.

*Check.* $18 + 24 = 42$, and $\frac{2}{3}$ of $18 = \frac{1}{2}$ of $24$ $(12 = 12)$.

**11.** The sum of two numbers is 60, and $\frac{1}{5}$ of the first is equal to $\frac{1}{7}$ of the second. Find the numbers.

**12.** A boy, on being asked his age, said, "$\frac{1}{5}$ of my age is 2 yr. more than $\frac{2}{3}$ of my age." Find his age.

**13.** If I multiply a certain number by 8, add 14 to the product, divide the sum by 5, and subtract the number itself from the quotient, the remainder is 10. Find the number.

**14.** What number is that which, if you multiply it by 3, subtract 8 from the product, divide the remainder by 5, and add 8 to the quotient, the sum will be the number itself?

**15.** Walter gave $\frac{1}{5}$ of his money for a pony, $\frac{3}{10}$ of the remainder for a jumper, and had \$63 left. How much had he at first?

**16.** The width of a rectangular field is $\frac{3}{10}$ of its length, and the distance around the field is 234 rd. Find the length and width of the field.

**17.** A father is now twice as old as his son, and 15 yr. ago the age of the son was $\frac{2}{7}$ of the age of the father. Find their ages.

Let $x$ = number of years in son's age,
then $2x$ = number of years in father's age.
$x - 15$ = number of years in son's age 15 yr. ago,
and $2x - 15$ = number of years in father's age 15 yr. ago.

$$\therefore x - 15 = \frac{2(2x - 15)}{7}. \qquad (1)$$

(1) $\times 7$, $7x - 105 = 4x - 30$.
Whence, $x = 25$. $\therefore$ 25 yr. is son's age.
$2x = 50$. $\therefore$ 50 yr. is father's age.

**18.** A's age is now $\frac{1}{4}$ of B's age, but 20 years from now A's age will be $\frac{1}{2}$ of B's age. Find their ages.

**19.** The ages of A and B are now 24 yr. and 42 yr., respectively. In how many years will the age of A be $\frac{5}{8}$ of the age of B?

**20.** Twelve years ago a man's age was $\frac{2}{5}$ of what it will be in twelve more years. Find his present age.

**21.** The sum of two numbers is 60, and the smaller is 12 more than $\frac{1}{5}$ of the greater. Find the numbers.

**22.** A has \$30 and B has \$54. How much must B give A so that A shall then have $\frac{3}{4}$ as much as B will have left?

**23.** A farmer sold some chickens and geese for $\$20\frac{3}{4}$, and there were 20 more chickens than geese. He received $\$1\frac{1}{4}$ each for the chickens, and $\$\frac{4}{5}$ each for the geese. How many of each did he sell?

**24.** Two numbers are to each other as 3 to 4. If 5 is subtracted from each, the smaller one will be $\frac{5}{7}$ of the larger. Find the two numbers.

HINT. Let $3x$ and $4x$ represent the numbers.

## FRACTIONAL EQUATIONS OF TWO UNKNOWN NUMBERS

**126.** 1. Solve $\frac{x}{2}+\frac{2y}{5}=13$ and $\frac{4x}{5}-\frac{3y}{10}=2$ for $x$ and $y$.

Multiply the 1st equation by 10, $5x+4y=130.$ (3)

Multiply the 2d equation by 10, $8x-3y=20.$ (4)

Now solve (3) and (4) as on page 41.

$$x=10,\ y=20.$$

First clear the equations of fractions, and then proceed according to the rules heretofore given.

Solve the following:

**1.** $\begin{cases}\frac{x}{2}+\frac{y}{3}=8,\\ \frac{x}{3}+\frac{y}{2}=7.\end{cases}$ **2.** $\begin{cases}\frac{1}{x}+\frac{1}{y}=\frac{1}{2},\\ \frac{1}{x}-\frac{1}{y}=\frac{1}{6}.\end{cases}$ **3.** $\begin{cases}\frac{3}{x}+\frac{5}{y}=\frac{8}{15},\\ \frac{9}{x}-\frac{22}{y}=\frac{3}{25}.\end{cases}$

**4.** $\begin{cases}\frac{x}{2}+\frac{y}{4}=8,\\ \frac{x}{4}-\frac{y}{8}=2.\end{cases}$ **5.** $\begin{cases}\frac{x}{4}+\frac{y}{6}=\frac{7}{2},\\ \frac{x}{3}-\frac{y}{8}=\frac{1}{2}.\end{cases}$ **6.** $\begin{cases}\frac{x+3}{y+3}=\frac{5}{6},\\ \frac{x-3}{y-3}=\frac{2}{3}.\end{cases}$

**7.** $\begin{cases}\frac{x}{2}+\frac{y}{3}=7,\\ \frac{x}{3}+\frac{y}{4}=5.\end{cases}$ **8.** $\begin{cases}\frac{x}{3}-\frac{y}{4}=5,\\ \frac{x}{4}-\frac{y}{6}=4.\end{cases}$ **9.** $\begin{cases}\frac{x}{7}+\frac{y}{5}=7,\\ \frac{x}{3}-\frac{y}{4}=2.\end{cases}$

**10.** $\begin{cases}\frac{5x}{14}-y=\frac{6}{7},\\ \frac{3y}{5}-\frac{x}{8}=\frac{1}{5}.\end{cases}$ **11.** $\begin{cases}\frac{x+3y-2}{3}=\frac{4y-x+3}{5},\\ \frac{3x-y+5}{10}=\frac{2x+5y+3}{12}.\end{cases}$

### PROBLEMS

1. A certain fraction is equal to $\frac{3}{4}$, and if 2 is added to each of its terms, it will then be equal to $\frac{4}{5}$. Find the fraction.

Let $x$ = the numerator, $y$ = the denominator. Then $\frac{x}{y}$ = the fraction. Hence, $\frac{x}{y}=\frac{3}{4}$, and $\frac{x+2}{y+2}=\frac{4}{5}$. Let the student solve these equations.

2. The sum of two fractions is $\frac{5}{6}$ and their difference is $\frac{1}{2}$. Find the fractions if their denominators are 9 and 30.

3. What fraction is that in which if the numerator is doubled and the denominator increased by 7, the value becomes $\frac{2}{5}$; but if the denominator is doubled and the numerator is increased by 2, the value becomes $\frac{3}{8}$?

4. The sum of two numbers is 51, and if the greater is divided by the less, the quotient will be 5 and the remainder 3. Find the numbers.

5. A father is now 24 yr. older than his son, and 6 years hence the age of the son will be $\frac{5}{9}$ of the age of the father. Find their present ages.

6. A and B together have \$200. If A gives B \$10, then $\frac{1}{11}$ of A's money will be equal to $\frac{1}{9}$ of B's. How much money has each?

7. A and B together can do a piece of work in 3 da. They can also do the work if A works 2 da. and B 6 da. Find the time it would take each alone to do the work.

8. A and B together can do a piece of work in 6 da., and $\frac{1}{4}$ of the work will be done if A works at it 1 da. and B 3 da. Find the time it will take each to do the work.

**9.** Five dry quarts and 10 liquid quarts equal 15 liters, and 7 dry quarts and 4 liquid quarts equal $11\frac{1}{2}$ liters. How many liters are there in a dry quart, and how many in a liquid quart?

**10.** The admission to an entertainment was $\$\frac{1}{2}$ for adults and $\$\frac{1}{4}$ for children. If the proceeds from 100 tickets amounted to $\$40\frac{3}{4}$, how many tickets of each kind were sold?

**11.** A boy bought 55 oranges, some at 8 for 35¢, and the remainder at 6 for 25¢, and the cost of all was $\$2.35$. How many oranges of each price did he buy?

**12.** The circumference of the hind wheel of a wagon is 55 in. more than that of the fore wheel, and the former makes as many revolutions in rolling 315 ft. as the latter does in rolling 238 ft. Find the number of inches in the circumference of each wheel.

**13.** A and B can do a piece of work in 4 da., and they can also do the work if A works 3 da. and B 8 da. In what time can each alone do the work?

**14.** A grocer has two kinds of sugar, one worth 5¢ and the other worth 6¢ a pound. How many pounds of each kind must be taken to form a mixture of 100 lb. worth $5\frac{3}{4}$¢ a pound?

**15.** A grocer bought a lot of corn at $\$\frac{1}{2}$ per bushel, and a lot of oats at $\$\frac{2}{5}$ per bushel, paying $\$64$ for the whole. After reserving 10 bu. oats and 20 bu. corn for his own use, he sold the remainder of the oats at $\$\frac{5}{8}$ per bushel, and the remainder of the corn at $\$\frac{3}{4}$ per bushel, and received for both $\$76\frac{1}{4}$. How many bushels of each did he buy?

### LITERAL EQUATIONS

**127. Literal equations** are equations in which some or all of the given numbers are represented by letters.

Find the value of $x$ in $ax - c = bx + d$.

Transpose, $ax - bx = c + d$.

Factor, $(a - b)\,x = c + d$.

Divide by $a - b$, $x = \dfrac{c + d}{a - b}$, *Ans.*

Sólve:

1. $2x = 6a$.
2. $cx = 5c^2$.
3. $nx = an + 5n$.
4. $6x = 4c$.
5. $a^2x = na^3$.
6. $bcx = ab - bc$.
7. $(n + 1)\,x = n^2 - 1$.
8. $x + ax = 1 - 3a - 4a^2$.
9. $(a + b)\,x = a^2 - b^2$.
10. $cx - 5x = c^2 - 3c - 10$.

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11. $ax - b^3 = a^3 - bx$.
12. $ax - 3a = a^3 + 2x - 14$.
13. $ax + b^3 = bx + a^3$.
14. $ax - bx = m - 2x + 3m$.
15. $\dfrac{ax}{2} + \dfrac{c}{3} = \dfrac{bx}{3} - \dfrac{d}{4}$.
16. $a + \dfrac{c}{x} = \dfrac{n}{x} - \dfrac{n - 2}{x}$.
17. $\dfrac{x}{a} - \dfrac{x}{b} + \dfrac{x}{c} = \dfrac{s}{abc}$.
18. $\dfrac{5x}{3} - \dfrac{ax}{4} - \dfrac{b}{8} = \dfrac{c - b}{6}$.

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19. $ax - 2x = a^3 - 2a^2 + 3a - 6$.
20. $ax + 3x = a^3 + 6a^2 + 11a + 6$.
21. $5cx + 1 - x = c(9 - 23c + 15c^2)$.

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22. $abx - c = ab$.
23. $p + ptx = a$.
24. $\dfrac{x}{a} + \dfrac{x}{b} + \dfrac{x}{c} = 1$.
25. $a - \dfrac{b}{1 - x} = c$.
26. $a - \dfrac{n - cx}{2} = n$.
27. $\dfrac{x - 2}{a} + \dfrac{x + 3}{b} = 1$.

**128. Solving an equation for any letter it contains.**

In equations like $x = a^2 - 6ab$ and $y = n^3 + rn^2$, the value of $x$ is said to be expressed in *terms of a and b;* and of $y$, in *terms of n and r.*

To solve an equation for any letter (or number) it may contain is to express the value of that letter in terms of all the other letters in the equation.

Solve $\frac{3+n}{5} - \frac{2-ma}{4} = 1$ for $a$.

Multiply by 20, $12 + 4n - 10 + 5ma = 20.$

Reduce, $4n + 5ma = 18.$

Transpose, $5ma = 18 - 4n.$

Divide by $5m$, $a = \frac{18-4n}{5m}$, *Ans.*

After clearing of fractions and reducing, all terms must be transposed to the second member except the one which contains the letter whose value is required.

Solve the following for the letters designated:

1. $3x + 5y = 12$ for $x$.
2. $9x + 7y = 25$ for $y$.
3. $4m - 3n = 10$ for $n$.
4. $7a - 2b = 16$ for $b$.
5. $\frac{3x}{4} + \frac{2x}{3} = a$ for $x$.
6. $\frac{4x}{5} - \frac{3y}{4} = b$ for $y$.
7. $2 = \frac{2m+n}{2+c}$ for $c$.
8. $\frac{2+a}{3} = \frac{2-c}{4}$ for $c$.
9. $\frac{2a+3b}{2} + \frac{4a-5b}{3} = 10$ for $b$.
10. $\frac{a+n}{a-n} + \frac{a-n}{a+n} = \frac{3}{a^2-n^2}$ for $a^2$.
11. $\frac{(a+b)^2}{3} - \frac{(a-b)^2}{3} = 2$ for $ab$.

### ELIMINATION BY COMPARISON

**129.** The equations $5x + 4y = 7$, (1)

$2x - 3y = 12$ (2)

were solved on page 41 by addition and subtraction. We will now solve them by another method called **elimination by comparison.**

Solve (1) for $x$, $x = \frac{7 - 4y}{5}$. (3)

Solve (2) for $x$, $x = \frac{12 + 3y}{2}$. (4)

Equate these values of $x$, $\frac{7 - 4y}{5} = \frac{12 + 3y}{2}$. (5)

Solve (5) for $y$, $y = -2$.

Substitute $-2$ for $y$ in (1), $x = 3$.

In like manner solve:

**1.** $\begin{cases} x + 5y = 11, \\ 3x + 2y = 7. \end{cases}$

**2.** $\begin{cases} 2x - 3y = -5, \\ 5x + 2y = 54. \end{cases}$

**3.** $\begin{cases} 8x - 15y = 36, \\ 21x - 20y = 172. \end{cases}$

**4.** $\begin{cases} 6x + 7y = 44, \\ 2x - 55y = -100. \end{cases}$

**5.** $\begin{cases} 7x + 11y = 68, \\ 9x - 4y = 33. \end{cases}$

**6.** $\begin{cases} 7x - 2y = 1, \\ 3x + 5y = 59. \end{cases}$

**7.** $\begin{cases} 15x + 7y = 29, \\ 9x + 15y = 39. \end{cases}$

**8.** $\begin{cases} 12x + 15y = 126, \\ 18x + 5y = 84. \end{cases}$

**9.** $\begin{cases} \frac{10}{x} - \frac{9}{y} = 4, \\ \frac{8}{x} - \frac{15}{y} = \frac{9}{2}. \end{cases}$

**10.** $\begin{cases} \frac{5}{3x} - \frac{6}{y} = -\frac{31}{12}. \\ \frac{3}{x} + \frac{5}{4y} = \frac{11}{8}. \end{cases}$

**11.** $\begin{cases} 2x + by = c, \\ ax + 3y = d. \end{cases}$

**12.** $\begin{cases} ax + by = 2a, \\ a^2x - b^2y = a^2 + b^2. \end{cases}$

### FORMULAS

**130.** A **formula** is a rule expressed in algebraic symbols. Formulas arise in solving general problems; that is, problems in which the known numbers are represented by letters. The answer to a problem of this character is a formula, and the *interpretation* of the formula is a rule by which all numerical problems of that kind may be solved.

### DISTANCE, TIME, AND RATE

**131.** A body moving at the rate of $r$ feet per second for $t$ seconds goes $d$ feet. Find $d$ in terms of $r$ and $t$.

Since the body moves $r$ feet in 1 second, in $t$ seconds it will go $t$ times $r$, or $tr$ feet.

But in $t$ seconds the body goes $d$ feet.

$$\therefore\ d = tr. \qquad (1)$$

The interpretation of this formula is:

*The distance is equal to the rate multiplied by the time.*

NOTE. By *distance* in these formulas is meant the *number* of units of length in the distance, by *time* the *number* of units of time, and by *rate* the *number* of units of length passed over in a unit of time. The student should remember that every letter represents merely a *number*.

Solve (1) for $t$, and give its interpretation.

*Ans.* $t = \frac{d}{r}$; *time equals distance divided by rate.*

Solve (1) for $r$, and give its interpretation.

Solve the following by the preceding formulas:

1. How far will a horse walk in $4\frac{1}{2}$ hr. at the rate of 6 mi. per hour?

2. In how many minutes will a bicyclist go 5280 yd. at the rate of 352 yd. per minute?

3. A train ran 147 mi. in $3\frac{1}{2}$ hr. Find its average rate.

NOTE. The formulas on the preceding page are to be employed in the solutions of the following problems.

1. A starts from Atlanta and rides toward Savannah at the rate of 7 mi. per hour. Four hours afterwards B starts from the same place and follows the same road at the rate of 9 mi. per hour. How far from Atlanta will he overtake A?

Let $x$ = the number of hours A travels.
Then $x - 4$ = the number of hours B travels,
$7x$ = the number of miles A travels, § 131,
and $9(x - 4)$ = the number of miles B travels, § 131.
Distance traveled by A = distance traveled by B.
$\therefore 7x = 9(x - 4)$,
or $7x = 9x - 36$.
$\therefore x = 18$, the number of hours A travels.
$7 \times 18 = 126$, the number of miles A travels, *Ans.*

2. A bicyclist who travels 8 mi. an hour is followed, after 3 hr., by a second bicyclist, who travels 12 mi. an hour. In how many hours will the second bicyclist overtake the first?

3. A man walked from the base to the top of a mountain and then back the same way to the base in 6 hours. In going up his rate was $1\frac{1}{2}$ mi. per hour and in returning, 3 mi. per hour. How far was it to the top of the mountain?

4. A train, whose rate is 30 mi. per hour, takes 1 hr. less time to run a certain distance than a train whose rate is 25 mi. per hour. Find the distance.

5. A leaves Jackson for Clinton the same time that B leaves Clinton for Jackson. In how many hours will they meet, if A could travel the whole distance in 12 hr. and B could travel the whole distance in 8 hr.?

## HANDS OF A CLOCK

NOTE. The following problems are solved by the formulas on page 111. Remember that the minute hand of a clock moves 12 times as fast as the hour hand.

In what follows, "spaces" mean minute spaces.

**1.** Find the time between 3 and 4 o'clock when the hands are together.

At 3 o'clock the hour hand is 15 spaces ahead of the minute hand, and therefore has 15 spaces less to go.

Let $x$ = the number of spaces the minute hand goes over.

Then $x - 15$ = the number of spaces the hour hand goes over.

$\therefore\ 12(x - 15) = x.$

$\therefore\ x = 16\frac{4}{11}.$

Hence, the time is $16\frac{4}{11}$ min. past 3 o'clock, or $3{:}16\frac{4}{11}$.

**2.** Find the time between 6 and 7 o'clock when the hands of a clock are together.

**3.** Find the time between 4 and 5 o'clock when the hands of a clock point in opposite directions.

HINT. The hour hand has $20 + 30$ spaces less to go than the minute hand.

**4.** Find the time between 5 and 5:30 o'clock when the hands of a clock will be perpendicular to each other.

HINT. The hour hand has $25 - 15$ spaces less to go than the minute hand.

**5.** Find the time between 5:30 and 6 o'clock when the hands of a clock will be perpendicular to each other.

**6.** Find the time between 9 and 10 o'clock when the hands of a clock point in opposite directions.

**7.** It is now between 10 and 11 o'clock, and the hands of a clock are perpendicular to each other. What time is it?

### SUM AND DIFFERENCE

**132.** The sum of two numbers is $s$ and their difference is $d$. Find the numbers.

Let $x$ = the greater number,

and $y$ = the smaller number.

Then $x + y = s$, (1)

and $x - y = d$. (2)

(1) + (2), $2x = s + d$. (3)

(3) ÷ 2, $x = \frac{s + d}{2}$. (4)

(1) − (2) $2y = s - d$. (5)

(5) ÷ 2, $y = \frac{s - d}{2}$. (6)

Formulas (4) and (6), being true for whatever numbers $s$ and $d$ stand for, are algebraic expressions of general rules for finding two numbers when their sum and difference are known. They may be interpreted thus:

*To find the greater number, add the difference and sum and take half of the result.*

*To find the smaller number, subtract the difference from the sum and take half of the result.*

Solve the following by the above formulas:

**1.** The sum of two numbers is 43 and their difference is 17. Find the numbers, and verify them.

**2.** James and John together have 215 apples and James has 27 more than John. How many has each?

**3.** In an election 30,275 votes were polled for two candidates A and B, and B was elected by a majority of 597 votes. How many votes did each receive?

**4.** Find $d$ in terms of $s$ and $x$, and interpret the result.

HINT. Solve (4) for $d$.

## VI. POWERS AND ROOTS

### POWERS

**133.** The product of equal numbers is a **power** of one of the equal numbers, § 55.

**134.** A power is indicated by an exponent, § 56.

Thus, $5^2 = 5 \times 5 = 25$, is the square or second power of 5.
Again, $5^3 = 5 \times 5 \times 5 = 125$, is the cube or third power of 5.

**135. Finding the powers of monomials.**

Find the value of $(-2a^2b)^3$; of $(-2a^2b)^4$.

(*a*) $-2a^2b \times -2a^2b \times -2a^2b = -8a^6b^3$.

(*b*) $-2a^2b \times -2a^2b \times -2a^2b \times -2a^2b = +16a^8b^4$.

*Raise the numerical coefficient to the required power; multiply the exponent of each literal factor by the exponent of the power to which it is to be raised.*

*All powers of a positive number are positive.*

*All even powers of a negative number are positive, and all odd powers are negative.*

Find the values of:

**1.** $(3xy)^2$. **2.** $(-2x^2y)^4$. **3.** $(-2a^3b^2)^7$.

**4.** $(5x^2y)^3$. **5.** $(-2x^2y)^5$. **6.** $(-3c^5x^3)^6$.

**7.** $(m^2n^6x)^5$. **8.** $(-a^2b^3c^4)^6$. **9.** $(-xy^2z^3)^{10}$.

**10.** $\left(\frac{2x^2}{3y^3}\right)^3$. $\frac{2x^2}{3y^3} \times \frac{2x^2}{3y^3} \times \frac{2x^2}{3y^3} = \frac{8x^6}{27y^9}$.

To raise a fraction to a power, raise both terms to the required power.

**11.** $\left(\frac{2a^3b}{5x^2y^3}\right)^3$. **12.** $\left(\frac{3x^2y^4}{4m^3}\right)^4$. **13.** $\left(-\frac{a^4b^2x}{2c}\right)^8$.

### SQUARE ROOT

**136.** The **square root** of an integer is one of the two equal factors of the integer.

Name the square roots of the following:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144.

**137.** A **perfect square** is a number that is the product of two equal integers or fractions; and an **imperfect square** is a number that is not the product of two equal integers or fractions.

Roots are indicated by the radical sign, $\sqrt{\ }$, to which the vinculum is attached, $\sqrt{\phantom{x}}$, to show how far the effect extends. For all roots except square roots, a small figure, called the **index**, is written in the opening, to indicate the particular root to be extracted.

Thus, $\sqrt{16\,a^4b^2}$, $\sqrt[3]{27\,a^6b^3}$, $\sqrt[4]{16\,a^4b^8}$, $\sqrt[5]{a^5b^{10}}$, etc.

**138. Finding the square roots of monomials.**

Find the square root of $49\,a^8b^6x^2$.

$\sqrt{49} = 7$, $\sqrt{a^8} = a^4$, $\sqrt{b^6} = b^3$, $\sqrt{x^2} = x$. $\therefore \sqrt{49\,a^8b^6x^2} = 7\,a^4b^3x$.

*Extract the square root of the coefficient, and divide the exponent of each literal factor by 2.*

Find the values of:

1. $\sqrt{25\,a^2}$.
2. $\sqrt{49\,a^4b^2x^6}$.
3. $\sqrt{121\,a^6b^4x^{12}}$.
4. $\sqrt{36\,b^8}$.
5. $\sqrt{144\,n^6y^{10}}$.
6. $\sqrt{100\,c^{14}d^{16}y^2}$.
7. $\sqrt{m^4x^2}$.
8. $\sqrt{m^{10}n^{12}x^8}$.
9. $\sqrt{169\,x^{16}y^{10}z^4}$.
10. $\sqrt{a^2(a+x)^4}$.
11. $\sqrt{4\,(a-x)^4(x+y)^6}$.
12. $\sqrt{9\,(a-c)^6}$.
13. $\sqrt{a^2(n+a)^8(x-y)^2}$.
14. $\sqrt{n^4(x-y)^8}$.
15. $\sqrt{9\,(x+a)^4(x-a)^{10}}$.

**139. Extracting the square roots of polynomials.**

Since the square of $t + o = t^2 + 2\,to + o^2$ (§ 79), the square-root of $t^2 + 2\,to + o^2$ is $t + o$.

Now we wish to determine how $t + o$ may be found when $t^2 + 2\,to + o^2$ is known, whatever may be the values of $t$ and $o$.

$$\begin{array}{r|l} t^2 + 2\,to + o^2 & t + o \\ \underline{t^2} & \\ 2\,t + o \;\big|\; 2\,to + o^2 & \\ \quad\;\big|\; \underline{2\,to + o^2} & \end{array}$$

Evidently, the first term, $t$, of the root is the square root of the first term, $t^2$, of the given number.

Subtracting $t^2$ from the given number leaves the remainder, $2\,to + o^2 = (2\,t + o)o$.

Now $2\,t$, *the double of the part of the root already found*, is called the **trial divisor.** Dividing the first term of the remainder, $2\,to$, by the trial divisor, gives the quotient $o$, the second term of the root. Annexing or adding $o$ to the trial divisor gives $2\,t + o$, which is called the **complete divisor.** The product of $2\,t + o$ by $o = 2\,to + o^2$, which exhausts the remainder.

Find the square root of $16\,x^2 - 40\,xy + 25\,y^2$.

$$\begin{array}{r|l} 16\,x^2 - 40\,xy + 25\,y^2 & 4\,x - 5\,y \\ \underline{16\,x^2} & \\ 8\,x - 5\,y \;\big|\; -40\,xy + 25\,y^2 & \\ \quad\;\big|\; \underline{-40\,xy + 25\,y^2} & \end{array}$$

In this example $4\,x$ and $-5\,y$ correspond to $t$ and $o$ in the typical example given above.

The square root of the first term is $4\,x$, which is the first term of the root. The square of $4\,x$ subtracted from the given number leaves the remainder, $-40\,xy + 25\,y^2$.

The double of $4\,x$ is $8\,x$, which is the trial divisor. Dividing $-40\,xy$ by $8\,x$ gives the quotient, $-5\,y$, which is the second term of the square root.

Annexing this new term, $-5\,y$, to the trial divisor gives the complete divisor, $8\,x - 5\,y$, which, when multiplied by $-5\,y$, exhausts the remainder.

Find the square roots of:

1. $25\,x^2 + 20\,xy + 4\,y^2$.
2. $9\,a^2 - 24\,ab + 16\,b^2$.
3. $36\,a^2 - 24\,ab + 4\,b^2$.
4. $a^2x^2 - 4\,axby + 4\,b^2y^2$.

Find the square roots of :

1. $x^2 - 6xy + 9y^2$.
2. $a^2x^2 - 10acx + 25c^2$.
3. $a^2 + 8ab + 16b^2$.
4. $16x^2 + 40a^2x + 25a^4$.
5. $x^2 - 12cx + 36c^2$.
6. $36b^4 - 84b^2n + 49n^2$.
7. $a^4 + 16a^2c + 64c^2$.
8. $9 + 72c^2n + 144c^4n^2$.
9. $a^6 - 10a^3c^2 + 25c^4$.
10. $169a^2x^4 - 26ax^2 + 1$.

The same method will apply to numbers of more than three terms if the trial divisor is obtained at each stage of the process by doubling the part of the root already found for a trial divisor, and annexing or adding the new term of the root to the trial divisor for a complete divisor.

11. Find the square root of

$$4a^4 - 12a^3b + 29a^2b^2 - 30ab^3 + 25b^4.$$

$$\begin{array}{r|l}
4a^4 - 12a^3b + 29a^2b^2 - 30ab^3 + 25b^4 & 2a^2 - 3ab + 5b^2 \\
4a^4 & \\
\hline
4a^2 - 3ab \;\big|\; -12a^3b + 29a^2b^2 & \\
\quad\quad\;\; -12a^3b + 9a^2b^2 & \\
\hline
4a^2 - 6ab + 5b^2 \;\big|\; 20a^2b^2 - 30ab^3 + 25b^4 & \\
\quad\quad\quad\quad\;\; 20a^2b^2 - 30ab^3 + 25b^4 & \\
\hline
\end{array}$$

The first two terms of the root are found as in the previous examples.

We now treat $2a^2 - 3ab$ as the first term of the root, and double it to obtain the next trial divisor, $4a^2 - 6ab$. Dividing the first term of the remainder, $20a^2b^2$, by the first term of this new trial divisor, gives the third term of the root, $5b^2$. Annexing $+5b^2$ to the trial divisor already found, gives the new complete divisor, $4a^2 - 6ab + 5b^2$. This, when multiplied by $5b^2$, gives the product, $20a^2b^2 - 30ab^3 + 25b^4$, which subtracted from the last remainder leaves zero. Therefore, the square root required is $2a^2 - 3ab + 5b^2$.

If there were other terms, we should continue in like manner until all the terms of the root were found.

Find the square roots of:

1. $4x^4+4x^3+5x^2+2x+1$.
2. $n^4+6n^3+11n^2+6n+1$.
3. $4y^4-16y^3+20y^2-8y+1$.
4. $16a^4-16a^3-12a^2+8a+4$.
5. $9x^4+6x^3-11x^2-4x+4$.
6. $9x^4-24x^3+28x^2-16x+4$.
7. $n^6+4n^5+4n^4+2n^3+4n^2+1$.
8. $9a^8+30a^6x+a^4x^2-40a^2x^3+16x^4$.
9. $3x^4-2x^5-x^2+2x+1+x^6$.

Arrange according to the descending powers of $x$.

$$
\begin{array}{rl|l}
 & x^6-2x^5+3x^4+0x^3-x^2+2x+1 & x^3-x^2+x+1 \\
\cline{3-3}
 & x^6 & \\
\cline{2-2}
2x^3-x^2 & -2x^5+3x^4 & \\
 & -2x^5+x^4 & \\
\cline{2-2}
2x^3-2x^2+x & 2x^4+0x^3-x^2 & \\
 & 2x^4-2x^3+x^2 & \\
\cline{2-2}
2x^3-2x^2+2x+1 & 2x^3-2x^2+2x+1 & \\
 & 2x^3-2x^2+2x+1 & \\
\cline{2-2}
\end{array}
$$

10. $4x^6+17x^2-22x^3+13x^4-24x-4x^5+16$.
11. $12a^2-4a+48a^4+1-64a^5+64a^6-4a-32a^3$.
12. $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$.
13. $\dfrac{x^2+4x+4}{a^2-6a+9}$.

$$\sqrt{\frac{x^2+4x+4}{a^2-6a+9}}=\frac{\sqrt{x^2+4x+4}}{\sqrt{a^2-6a+9}}=\frac{x+2}{a-3}.$$

14. $\dfrac{x^4+2x^3+3x^2+2x+1}{x^4+4x^3+6x^2+4x+1}$.
15. $\dfrac{x^4-2x^2y+2x^2z^2-2yz^2+y^2+z^4}{x^2+2xy+y^2-2x-2y+1}$.

**140. Finding the square roots of arithmetical numbers.**

To extract the square root of an arithmetical number we first determine how many digits or figures there will be in the root, and what the first digit will be, which we can do by inspection as follows. Note that each number is divided into periods of two figures each, beginning at the right.

Since 729 lies between 400 and 900, its square root lies between 20 and 30. Hence, there are two figures in the square root of 7'29, and the first figure, 2, is the square root of the greatest square contained in the first period, 7.

Since 403,225 lies between 360,000 and 490,000, its square root lies between 600 and 700. Hence, there are three figures in the square root of 40'32'25, and the first figure, 6, is the square root of the greatest square contained in the first period, 40.

Compare the number of periods in the following squares and the number of figures in the square roots:

| Number | Root | Number | Root | Number | Root |
|---|---|---|---|---|---|
| 1 | 1 | 1'00 | 10 | 1'00'00 | 100 |
| 81 | 9 | 98'01 | 99 | 99'80'01 | 999 |

*There are as many figures in the square root of a perfect square as there are 2-figure periods in the number.*

*The first figure of the root is the square root of the greatest square contained in the left-hand period.*

Note. The left-hand period may consist of only one figure.

Give the first figure, and the number of figures in the square roots of:

| | | |
|---|---|---|
| 1. 6'25. | 2. 5'76. | 3. 30'25. |
| 4. 51'84. | 5. 45'15'84. | 6. 84'27'24. |
| 7. 1'76'89. | 8. 17'30'56'00. | 9. 52'27'29'00. |

Find the square root of 6'76.

As there are two periods in 6'76, there will be two figures in the square root of 676, and the tens' figure will be 2.

Denoting the tens of the root, 20, by $t$ and the ones by $o$, we have $(t + o)^2 = t^2 + 2\,to + o^2 = 676.$

Hence, the following operation is like the typical one given on page 117.

$$\begin{array}{rrl} & & t + o \\ & & 6'76 \mid 20 + 6 \\ & & \underline{4\ 00} \\ \text{Trial divisor,} & 2t = 40 & 2\ 76 \\ & o = \underline{\ 6} & \\ \text{Complete divisor,} & 2t + o = 46 & \underline{2\ 76} \end{array}$$

Therefore, the square root of 676 = 20 + 6 = 26.

In like manner find the square roots of:

| | | |
|---|---|---|
| **1.** 256. | **2.** 1024. | **3.** 2916. |
| **4.** 324. | **5.** 1296. | **6.** 7225. |
| **7.** 625. | **8.** 2209. | **9.** 9216. |

NOTE. When some familiarity with the above process has been gained, the unnecessary ciphers may be omitted, as shown below in extracting the square roots of 3025 and 8649.

$$\begin{array}{rl} & 30'25 \mid 55 \\ & \underline{25} \\ 105 & \underline{5\ 25} \\ & \underline{5\ 25} \end{array} \qquad \begin{array}{rl} & 86'49 \mid 93 \\ & \underline{81} \\ 183 & \underline{5\ 49} \\ & \underline{5\ 49} \end{array}$$

Find the square roots of:

| | | |
|---|---|---|
| **10.** 6241. | **11.** 6561. | **12.** 5329. |
| **13.** 4356. | **14.** 2916. | **15.** 9025. |

The square roots of larger numbers may be found in like manner by regarding, at any stage of the process, the part of the root already found as *tens* and the figure sought as *ones*.

Extract the square roots of 132,496 and 965.9664.

| | | | |
|---|---|---|---|
| 13'24'96 | 364 | 9'65.96'64 | 31.08 |
| 9 | | 9 | |
| 66 \| 4 24 | | 61 \| 65 | |
| 3 96 | | 61 | |
| 724 \| 28 96 | | 6208 \| 4 96 64 | |
| 28 96 | | 4 96 64 | |

NOTES. 1. Decimals are pointed off into periods by beginning with tenths and passing to the right, annexing a cipher if necessary to complete a whole number of decimal periods.

2. If any dividend is less than the divisor, write a 0 in the root, and also on the right of the divisor, and annex the next period to the dividend for a new dividend.

3. If there is a remainder after all the periods have been used, annex periods of decimal 0's, and extend the work to any desired degree of exactness.

4. To find the square root of a fraction whose terms are perfect squares, extract the square roots of the numerator and denominator for the terms of the required root; if the terms are not perfect squares, reduce the fraction to a decimal, and then extract the square root.

Find the square roots of:

| | | |
|---|---|---|
| 1. 15,625. | 2. 522,729. | 3. 5,764,801. |
| 4. 59,049. | 5. 297,025. | 6. .000729. |
| 7. 46,656. | 8. 117,649. | 9. 170.0416. |
| 10. 20,164. | 11. 390,625. | 12. 7,728,400. |
| 13. 52,441. | 14. 537,289. | 15. .00021025. |
| 16. 18,769. | 17. 328,329. | 18. .00459684. |
| 19. 17,424. | 20. 315,844. | 21. 36.8449. |
| 22. 20,449. | 23. 667,489. | 24. .00002209. |

Find the value of:

25. $\sqrt{2}$. 26. $\sqrt{\frac{7}{8}}$. 27. $\sqrt{1442\frac{1}{4}}$. 28. $\sqrt{85{,}059\frac{4}{9}}$.

29. $\sqrt{3}$. 30. $\sqrt{\frac{4}{5}}$. 31. $\sqrt{2500\frac{16}{25}}$. 32. $\sqrt{4900\frac{9}{49}}$.

## VII. QUADRATIC EQUATIONS

### DEFINITIONS

**141.** A **quadratic equation** of one unknown number is an equation of the form $x^2 + m = 0$ (a **pure quadratic**) or $x^2 + nx + r = 0$ (an **affected quadratic**), or one which may be reduced to one of these forms, $m$, $n$, and $r$ being known numbers and $x$ an unknown number.

Thus, $x^2 - 9 = 0$ is a pure quadratic; so also is $(x + 4)(x - 4) = 0$, since it may be reduced to $x^2 - 16 = 0$.

Again, $x^2 - 6x + 7 = 0$ is an affected quadratic; so also is $(x + 7)(x - 4) = 0$, since it may be reduced to $x^2 + 3x - 28 = 0$.

A pure quadratic in its simplest form has only two terms, one being the second power of the unknown number and the other a known number.

An affected quadratic in its simplest form has only three terms, viz.: the second power of the unknown number, the first power of the unknown number multiplied by a known number, and a known number.

**142.** A **root** of a quadratic equation is the number which when substituted for the unknown number will satisfy the equation.

**143.** To **solve** a quadratic equation is to find its roots.

**144. Solving quadratic equations that are already factored.**

Solve $(x - 3)(x + 5) = 0$.

($a$) If $x - 3 = 0$, the equation will be satisfied. $\therefore x = 3$.

($b$) If $x + 5 = 0$, the equation will be satisfied. $\therefore x = -5$.

PROOF. Substitute 3 for $x$, $(3 - 3)(3 + 5) = 0 \times 8 = 0$.

Substitute $-5$ for $x$, $(-5 - 3)(-5 + 5) = -8 \times 0 = 0$.

Solve and verify the following:

1. $(x+2)(x-2)=0.$
2. $(x-8)(x+3)=0.$
3. $(x-5)(x+5)=0.$
4. $(x-5)(x-5)=0.$
5. $(x+8)(x-8)=0.$
6. $(x+9)(x-4)=0.$
7. $(x-9)(x+9)=0.$
8. $(x+5)(x+7)=0.$
9. $(x-1)(x+1)=0.$
10. $(x-8)(x+5)=0.$

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11. $(x+a)(x-a)=0.$
12. $(x+c)(x-a)=0.$
13. $(x-n)(x+n)=0.$
14. $(x+a)(x+b)=0.$
15. $(x-c)(x+c)=0.$
16. $(x-n)(x+r)=0.$
17. $(x+b)(x-b)=0.$
18. $(x-d)(x-e)=0.$

---

19. $(2x-5)(2x+5)=0.$
20. $(8x+2)(2x-3)=0.$
21. $(2x+1)(2x-1)=0.$
22. $(6x-3)(4x+2)=0.$
23. $(6x+4)(6x-4)=0.$
24. $(3x+4)(4x-3)=0.$
25. $(5x-3)(5x+3)=0.$
26. $(4x-6)(6x-5)=0.$

In like manner, any equation may be solved when it is expressed in terms of its factors, and there will be as many roots as there are factors of the first degree.

27. $(x-1)(x-2)(x-3)=0.$
28. $(x+5)(x-4)(x+2)=0.$
29. $(x-3)(x+3)(x-6)=0.$
30. $(3x-2)(2x+3)(x+8)=0.$
31. $(4x-8)(6x+2)(8x-12)=0.$
32. $(x+5)(2x-3)(x-9)(5x+6)=0.$
33. $(x-4)(x+4)(3x+6)(7x-8)=0.$

**145. Solving quadratics by factoring.**

Solve $x^2 - 64 = 0$ and $x^2 - 3x - 40 = 0$.

$x^2 - 64 = (x + 8)(x - 8)$.
$x + 8 = 0$. $\therefore x = -8$.
$x - 8 = 0$. $\therefore x = +8$.

$x^2 - 3x - 40 = (x - 8)(x + 5)$.
$x - 8 = 0$. $\therefore x = 8$.
$x + 5 = 0$. $\therefore x = -5$.

Solve:

1. $x^2 - 16 = 0$.
2. $x^2 + 2x - 35 = 0$.
3. $x^2 - 81 = 0$.
4. $x^2 - 8x + 15 = 0$.
5. $x^2 - 121 = 0$.
6. $x^2 + 9x + 18 = 0$.
7. $x^2 - 49a^2 = 0$.
8. $x^2 - 4x - 45 = 0$.
9. $x^2 - 64n^4 = 0$.
10. $x^2 + 7x - 60 = 0$.

---

11. $4x^2 - 25a^2 = 0$.
12. $x^2 - 3ax - 70a^2 = 0$.
13. $9x^2 - 16c^2 = 0$.
14. $x^2 + 4bx - 96b^2 = 0$.
15. $25x^2 - m^2 = 0$.
16. $x^2 - 8cx + 12c^2 = 0$.
17. $144x^2 - 1 = 0$.
18. $x^2 + 9dx + 20d^2 = 0$.
19. $81x^2 - a^2 = 0$.
20. $x^2 - mx - 72m^2 = 0$.

Any equation that can be expressed in terms of its simple factors may be solved in this manner.

21. $x^3 + 2x^2 - 9x - 18 = 0$.

$x^3 + 2x^2 - 9x - 18 = x^2(x + 2) - 9(x + 2)$
$= (x^2 - 9)(x + 2)$
$= (x + 3)(x - 3)(x + 2)$
$x + 3 = 0,\ x - 3 = 0,\ x + 2 = 0.\ \therefore x = -3,\ x = +3,\ x = -2.$

22. $x^3 - 3x^2 - x + 3 = 0$.
23. $x^3 + 5x^2 - 16x - 80 = 0$.
24. $2x^3 - 7x^2 - 8x + 28 = 0$.

**146.** The general solution of quadratic equations depends on extracting the square roots of numbers. Hence, the following facts about square roots should be carefully noted:

(*a*) *The square root of a number has the sign* $\pm$.

This sign means "+ or −," or "+ and −," and is so read.

Thus, the square root of 16 is $\pm 4$, because $+4 \times +4 = 16$ and $-4 \times -4 = 16$.

(*b*) *The square root of an imperfect square cannot be found exactly.*

Thus, $\sqrt{2}$ lies between 1.4142 and 1.4143 and also between $-1.4142$ and $-1.4143$.

It can, however, be determined approximately to any required degree of accuracy.

(*c*) *A negative number has no square root.*

Thus, $-25$ has no square root, because $+5 \times +5 = +25$, and $-5 \times -5 = +25$. No sign multiplied by itself gives $-$.

The indicated square root of a negative number is called an **imaginary number.**

Thus, $\sqrt{-9}$, $\sqrt{-7}$, $\sqrt{-a}$, are imaginary numbers.

(*d*) *The square roots of the members of an equation are equal to each other.*

Thus, if $4x^2 = 36$, then $\sqrt{4x^2} = \sqrt{36}$, or $2x = \pm 6$.

Again, if $(x+2)^2 = a^2 - 2ab + b^2$,

then $\sqrt{(x+2)^2} = \sqrt{a^2 - 2ab + b^2}$;

that is, $x + 2 = \pm (a - b)$,

which means $x + 2 = +a - b$

and $x + 2 = -a + b$.

**147. Solving pure quadratic equations.**

Solve the equation $2x^2 - 8 = \frac{x^2}{3} + 7$.

| | | |
|---|---|---|
| Given equation, | $2x^2 - 8 = \frac{x^2}{3} + 7.$ | (1) |
| Multiply (1) by 3, | $6x^2 - 24 = x^2 + 21.$ | (2) |
| Transpose, | $6x^2 - x^2 = 24 + 21.$ | (3) |
| Reduce (3), | $5x^2 = 45.$ | (4) |
| Divide (4) by 5, | $x^2 = 9.$ | |
| Extract the square root, | $x = \pm 3.$ | |

Find both values of $x$ in:

1. $x^2 + 7 = 23.$
2. $5x^2 - 2 = 2x^2 + 25.$
3. $x^2 - 5 = 44.$
4. $2x^2 + 27 = 32 - 3x^2.$
5. $x^2 + 75 = 4x^2.$
6. $x(2x - 3) = x^2 - 3x + 9.$
7. $7x^2 - 24 = x^2.$
8. $(x + 3)(x - 3) = 72.$
9. $\frac{x^2 + 2}{3} + 16 = x^2.$
10. $\frac{x - 3}{7} = \frac{13}{x + 3}.$
11. $x^2 - 9 = n^2 - 6n.$
12. $(x - a)(x + a) = c^2 + 2ac.$

13. The product of the third by the fourth of a number is 108. Find the number.

14. When a certain number is divided by 5, the quotient is the same as the quotient of 20 by the number. Find the number.

15. Two numbers are to each other as 3 to 4, and the difference of their squares is 63. Find the numbers.

HINT. Let $3x$ and $4x$ represent the numbers.

16. Two numbers are to each other as 2 to 3, and their product is 150. Find the numbers.

17. Two numbers are to each other as 5 to 12, and the sum of their squares is 676. Find the numbers.

**148. Completing the square.**

Find the square root of $x^2 - 10x + 25$.

Is $x^2 + 8x + 16$ a perfect square?

What number must be added to $x^2 + 12x$ to form a square? Complete the square, $x^2 - 16x + (\ )$.

**Completing the square** is adding such a number to a binomial of the form $x^2 + mx$ as will make the trinomial thus formed a perfect square.

Complete the square $x^2 + 20x + (\ )$.

According to § 96 the number to be added is the square of half the coefficient of $x$. Hence, the required square is $x^2 + 20x + (\frac{20}{2})^2$, or $x^2 + 20x + 100$.

Complete the squares:

| | | |
|---|---|---|
| 1. $x^2 + 4x + (\ )$. | 2. $x^2 + 5x$. | 3. $x^2 - 2mx$. |
| 4. $x^2 - 6x + (\ )$. | 5. $x^2 + 9x$. | 6. $x^2 + 3ax$. |
| 7. $x^2 + x + (\ )$. | 8. $x^2 - 11x$. | 9. $x^2 - 6ax$. |
| 10. $x^2 - x + (\ )$. | 11. $x^2 - 15x$. | 12. $x^2 + 9nx$. |
| 13. $x^2 - \frac{2}{3}x + (\ )$. | 14. $x^2 + \frac{3}{4}x$. | 15. $x^2 + \frac{2a}{5}x$. |
| 16. $x^2 - \frac{6}{5}x + (\ )$. | 17. $x^2 - \frac{5}{2}x$. | 18. $x^2 + \frac{2n}{m}x$. |

Find the square roots of the following after completing the squares:

| | | |
|---|---|---|
| 19. $x^2 - 6x$. | 20. $x^2 + 9x$. | 21. $x^2 - 2cx$. |
| 22. $x^2 - 8x$. | 23. $x^2 - 7x$. | 24. $x^2 + 3ax$. |
| 25. $x^2 - \frac{2}{5}x$. | 26. $x^2 + \frac{3}{8}x$. | 27. $x^2 + \frac{2b}{c}x$. |
| 28. $x^2 + \frac{6}{7}x$. | 29. $x^2 - \frac{5}{9}x$. | 30. $x^2 - \frac{m}{n}x$. |

**149. Solving affected quadratics.**

Solve the equation $3x^2 - 36x = -81$.

| | | |
|---|---|---|
| Given equation, | $3x^2 - 36x = -81.$ | (1) |
| Divide (1) by 3, | $x^2 - 12x = -27.$ | (2) |
| Complete the square, | $x^2 - 12x + 36 = -27 + 36 = 9.$ | (3) |
| Extract square root of (3), | $x - 6 = \pm 3.$ | (4) |
| Solve (4), using upper sign, | $x = 6 + 3 = 9.$ | |
| Solve (4), using lower sign, | $x = 6 - 3 = 3.$ | |

NOTE. In (3) 36 is added to the first member for the purpose of completing the square, and to the second member in order to preserve the equality.

Verify by substituting first 9 and then 3 for $x$ in equation (2).

| | |
|---|---|
| $9^2 - 12(9) = -27.$ | $3^2 - 12(3) = -27.$ |
| $81 - 108 = -27.$ | $9 - 36 = -27.$ |
| $-27 = -27.$ | $-27 = -27.$ |

Solve the equation $x^2 + 3x - 28 = 0$.

| | | |
|---|---|---|
| Transpose 28, | $x^2 + 3x = 28.$ | (1) |
| Complete square, | $x^2 + 3x + \frac{9}{4} = 28 + \frac{9}{4} = 12\frac{1}{4}.$ | (2) |
| Extract square root, | $x + \frac{3}{2} = \pm 3\frac{1}{2}.$ | (3) |
| Solve (3), using upper sign, | $x = -\frac{3}{2} + 3\frac{1}{2} = 4.$ | |
| Solve (3), using lower sign, | $x = -\frac{3}{2} - 3\frac{1}{2} = -7.$ | |

Verify by substituting first $+4$ and then $-7$ for $x$ in the original equation.

| | |
|---|---|
| $(+4)^2 + 3(+4) - 28 = 0.$ | $(-7)^2 + 3(-7) - 28 = 0.$ |
| $16 + 12 - 28 = 0.$ | $49 - 21 - 28 = 0.$ |
| $0 = 0.$ | $0 = 0.$ |

NOTES. 1. The student should be required to verify every solution for both values of $x$. It is not only a useful exercise in the manipulation of numbers, but it serves to impress the student more and more with the fundamental fact that the roots of an equation are the numbers that satisfy the equation.

2. When practical, the equations should be solved also by factoring, thus carrying on the two methods side by side.

Solve the following:

1. $x^2 - 6x = -8$.
2. $x^2 + 5x - 14 = 0$.
3. $x^2 + 8x = -7$.
4. $x^2 - 3x - 54 = 0$.
5. $x^2 - 4x = 12$.
6. $x^2 + x - 72 = 0$.
7. $x^2 + 2x = 35$.
8. $x^2 - 7x + 12 = 0$.
9. $x^2 - 8x = 20$.
10. $x^2 + 9x + 20 = 0$.

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11. $x^2 - 14x = -40$.
12. $x^2 - 15x + 56 = 0$.
13. $x^2 + 18x = -72$.
14. $x^2 + 25x + 100 = 0$.
15. $x^2 - 26x = 120$.
16. $x^2 - 15x - 126 = 0$.
17. $x^2 - 34x = 240$.
18. $x^2 + 27x - 280 = 0$.
19. $x^2 + 40x = 500$.
20. $x^2 - 43x - 350 = 0$.

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21. $5x^2 - 3x - 2 = 0$.

| | | |
|---|---|---|
| Transpose 2, | $5x^2 - 3x = 2$. | (1) |
| Divide (1) by 5, | $x^2 - \frac{3}{5}x = \frac{2}{5}$. | (2) |
| Complete square, | $x^2 - \frac{3}{5}x + \frac{9}{100} = \frac{2}{5} + \frac{9}{100} = \frac{49}{100}$. | (3) |
| Extract square root, | $x - \frac{3}{10} = \pm\frac{7}{10}$. | (4) |
| Solve (4), using upper sign, | $x = \frac{3}{10} + \frac{7}{10} = 1$. | |
| Solve (4), using lower sign, | $x = \frac{3}{10} - \frac{7}{10} = -\frac{2}{5}$. | |

22. $5x^2 - 4x - 1 = 0$.
23. $4x^2 - 9x = 9$.
24. $3x^2 + 2x - 5 = 0$.
25. $6x^2 + 5x = 34$.
26. $7x^2 - 8x - 12 = 0$.
27. $6x^2 + x = 15$.
28. $9x^2 - 8x - 17 = 0$.
29. $4x^2 - 3x = 52$.

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30. $x^2 - 2ax = -a^2 + b^2$.
31. $x^2 - 2mx = 35m^2$.
32. $x^2 - 2ax = 2ab + b^2$.
33. $x^2 + 3ax = 70a^2$.
34. $\frac{4}{9}x^2 - 4ax = b^2 - a^2$.
35. $10x^2 - cx = 3c^2$.
36. $x^2 - 2(2a - c)x = 3c^2 + 8ac - 3a^2$.

Solve the equation $\frac{4x}{7} - \frac{2x^2}{3} = \frac{10x}{3} - \frac{20}{7}$.

Multiply the given equation by 21, and we have

$$12x - 14x^2 = 70x - 60. \quad (1)$$

Reduce (1), change sign, $14x^2 + 58x = 60.$ (2)

Divide (2) by 14, $x^2 + \frac{29}{7}x = \frac{30}{7}.$

Complete square, $x^2 + \frac{29}{7}x + \frac{841}{196} = \frac{30}{7} + \frac{841}{196} = \frac{1681}{196}.$

Extract square root, $x + \frac{29}{14} = \pm \frac{41}{14}.$

Whence, $x = \frac{6}{7}$ or $-5.$

Solve the following:

1. $\frac{2x^2}{3} - \frac{5x}{2} = \frac{2}{3}.$
2. $\frac{3x^2}{2} + x - \frac{4}{3} = 0.$
3. $\frac{2}{3} - \frac{x^2}{4} = \frac{x}{6}.$
4. $\frac{4x^2}{3} - \frac{3x}{2} + \frac{1}{6} = 0.$
5. $\frac{x}{3} - 5 = -\frac{18}{x}.$
6. $\frac{x}{6} + 1 + \frac{4}{3x} = 0.$
7. $\frac{x}{5} - \frac{5}{3} = -\frac{10}{3x}.$
8. $\frac{x}{9} - \frac{5}{3} + \frac{4}{x} = 0.$
9. $\frac{x - 30}{4} = \frac{16}{x}.$
10. $\frac{x + 10}{7} = \frac{8}{x}.$
11. $\frac{x}{5} - \frac{8}{x + 6} = 0.$

HINT. Clear of fractions by multiplying by $5(x + 6)$.

12. $\frac{x}{3} - \frac{5}{x - 2} = 0.$
13. $\frac{3}{x} - \frac{x + 4}{4} = 0.$
14. $\frac{x}{5} - \frac{11}{3x - 4} = 0.$
15. $\frac{x}{7} - \frac{3}{2x + 1} = 0.$
16. $\frac{x}{7} - \frac{10}{3x - 1} = 0.$
17. $\frac{x}{x + 8} = \frac{x + 3}{2x + 1}.$

HINT. Clear of fractions by multiplying by $(x + 8)(2x + 1)$.

18. $\frac{2x - 2}{5x + 5} = \frac{x^2 - 1}{x + 1}.$
19. $\frac{x + 3}{x} + \frac{7x}{x + 3} = \frac{23}{4}.$

### PROBLEMS INVOLVING QUADRATICS

**1.** The difference of two numbers is 5 and their product is 84. Find the numbers.

Let $x$ = the smaller number.
Then $x + 5$ = the larger number,
and $x(x + 5)$ = their product.
But 84 = their product also.
$\therefore x(x + 5) = 84$.
Whence, $x = 7$, or $-12$, the smaller number.
$x + 5 = 12$, or $-7$, the larger number.

NOTE. While both sets of these answers satisfy the conditions algebraically, yet in practical problems the negative answers are usually not considered. They are omitted in the problems that follow.

**2.** The product of two consecutive integers is 56. Find the integers.

HINT. Let $x$ and $x + 1$ = the integers, respectively.

**3.** The sum of the squares of two consecutive integers is 145. Find the integers.

**4.** Find the number whose square diminished by 3 times the number itself leaves a remainder of 54.

**5.** The square of a number increased by 5 times the number itself is 104. Find the number.

**6.** The sum of two numbers is 7 and the sum of their squares is 25. Find the numbers.

HINT. Let $x$ = one of the numbers, then $7 - x$ = the other.

**7.** The difference of two numbers is 3 and the sum of their squares is 117. Find the numbers.

**8.** Find two numbers whose difference is 7, and whose sum multiplied by the greater number is 345.

**9.** A number of sheep cost \$60. If 3 more sheep had been bought for the same sum, each sheep would have cost \$1 less. How many sheep were bought?

Let $x$ = the number bought.

Then $\frac{60}{x}$ = the number of dollars each cost,

and $\frac{60}{x+3}$ = the number of dollars each would have cost had there been 3 more.

$$\therefore \frac{60}{x} - \frac{60}{x+3} = 1.$$

Multiply by $x(x+3)$, reduce, $x^2+3x = 180$.

Whence, $x = 12$.

**10.** I bought some flour for \$96. If I had obtained 8 bbl. more for the same money, the price per barrel would have been \$2 less. How many barrels did I buy?

**11.** A train ran 120 mi. in a certain number of hours, and the number of hours would have been 1 less had the rate of the train per hour been 10 mi. more. Find the rate of the train per hour.

**12.** An estate of \$12,600 was left to a number of sons. If there had been 2 sons less, the sum received by each would have been \$1050 more. Find the number of sons.

**13.** A clothier, having bought some cloth for \$30, found that if he had received 3 yd. more for the same money, the cloth would have cost him \$$\frac{1}{2}$ less per yard. How many yards did he buy?

**14.** A farmer purchased some hogs for \$168, and later sold all but 4 of them for the same money. If his profit on each hog sold was \$1, how many hogs did he buy?

**15.** A bar of iron weighs 36 lb. If the bar were 1 ft. longer, the weight per foot would be $\frac{1}{2}$ of a pound less Find the length of the bar.

### AREAS OF RECTANGLES

**150.** If the length and width of a rectangle are $l$ and $b$ linear units, respectively, and the number of square units in its surface is $a$, it is shown in geometry that

$$a = b \times l \qquad \text{(A)}$$

1. Find the area or number of square inches in a rectangle whose length is 24 in. and whose width is $9\frac{1}{2}$ in.

2. Solve (A) for $l$, and give the interpretation.

3. Find the length of a rectangular field whose area is 60 A. and whose width is 80 rd.

4. The length of a rectangular field is 3 ch. more than its width, and the area is 4 A. Find the sides.

5. The perimeter of a rectangle is 20 in. and its area is 24 sq. in. Find its sides.

6. The area of a triangle whose base is $b$ and altitude $h$ is one half the area of a rectangle whose length is $b$ and width $h$. Find the formula for the area of a triangle, and give its interpretation.

7. The base of a triangle is 121 yd. and its area is 2 A. Find its altitude.

8. The area of a square is doubled by increasing its length by 15 in. and decreasing its width by 2 in. Find its sides.

9. A plot of ground is 18 by 12 ft., around which is a walk of uniform width, whose area is equal to that of the plot. Find the width of the walk.

# PART TWO — HIGH SCHOOL ALGEBRA

## VIII. NUMBERS, OPERATIONS, AND RELATIONS

(Review and Extensions)

### FUNDAMENTAL OPERATIONS

**151.** A **fundamental operation** is finding one of three related numbers when the other two are known.

There are seven fundamental operations, viz. ($a$ and $b$ being known numbers):

1. If $a + b = x$, finding $x$ is *addition*.
2. If $a + x = b$, finding $x$ is *subtraction*.
3. If $a \times b = x$, finding $x$ is *multiplication*.
4. If $a \times x = b$, finding $x$ is *division*.
5. If $a^b = x$, finding $x$ is *involution*.
6. If $x^a = b$, finding $x$ is *evolution*.
7. If $a^x = b$, finding $x$ is *finding a logarithm*.

NOTE. This book does not treat of logarithms.

**152.** Two operations are **inverse** when either has the effect of undoing what was done by the other.

Subtraction is the inverse of addition.

Thus, if $a$ plus $b = s$, then $s$ less $b = a$.

Division is the inverse of multiplication.

Thus, if $b$ times $a = p$, then $p$ divided by $b = a$.

Evolution is the inverse of involution.

Thus, if $a^b = p$, then $\sqrt[b]{p} = a$.

**153. Equivalent expressions** are different ways of expressing the same fact or relation. The following are pairs of equivalent expressions, the first involving the symbols of the direct operations and the second the symbols of the inverse operations:

(1) $a + b = s$ and $a = s - b$;

(2) $l \times k = p$ and $l = p \div k$;

(3) $n^r = m$ and $n = \sqrt[r]{m}$.

### FUNDAMENTAL DEFINITIONS

**154.** Whether $a$ is greater than, equal to, or less than $b$, $a - b$ is *defined* as a number such that when increased by $b$ the sum is $a$.

That is, symbolically, $(a - b) + b = a$.

By this definition *zero* and *negative numbers* are added to the system of numbers.

Thus, $5 - 5$ or 0 is a number such that $0 + 5 = 5$; $2 - 5$ or $-3$ is a number such that $-3 + 5 = 2$.

**155.** Whether $a$ is exactly divisible by $b$ or not, $a \div b$ is *defined* as a number such that when multiplied by $b$ the product is $a$.

That is, symbolically, $(a \div b) \times b = a$.

By this definition *fractions* are added to the system of numbers.

Thus, $\frac{7}{3}$ is a number such that $\frac{7}{3} \times 3 = 7$.

**156.** Whether $p$ is an exact $r$th power or not, $\sqrt[r]{p}$ is *defined* as a number such that when raised to the $r$th power, the result is $p$.

That is, symbolically, $(\sqrt[r]{p})^r = p$.

By this definition *irrationals* and *imaginaries* are added to the system of numbers.

Thus, $\sqrt{2}$ is a number such that $(\sqrt{2})^2 = 2$;

$\sqrt{-2}$ is a number such that $(\sqrt{-2})^2 = -2$.

**157.** DEFINITION. If a number is not the $n$th power of an integer or a fraction, then its $n$th root, if not imaginary, § 225, is an **irrational number** or an **irrational.***

Thus, $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[5]{7}$ are irrational numbers, whereas $\sqrt{9}$, $\sqrt[3]{125}$, $\sqrt[5]{32}$ are rational numbers.

**158. Facts about irrationals.** (1) *An irrational is not an integer.*

For simplicity, consider the irrational $\sqrt{2}$. By definition, $(\sqrt{2})^2 = 2$. Now $1^2 = 1$ and $2^2 = 4$. Hence, $\sqrt{2}$ lies between the integers 1 and 2, and is therefore not an integer.

(2) *An irrational is not a fraction.*

For, let $\sqrt{2} = \frac{m}{n}$, a fraction in its lowest terms. Then $(\sqrt{2})^2 = \left(\frac{m}{n}\right)^2$, or $2 = \frac{m^2}{n^2}$, which is impossible; because $m^2$ cannot contain $n^2$ exactly 2 times unless $m$ and $n$ have a common factor. Therefore $\sqrt{2}$ is not a fraction.

(3) *An irrational may be expressed approximately in terms of integers and fractions to any desired degree of accuracy.*

Thus, by the method given on page 122 it may be shown that $\sqrt{2}$ is greater than 1.414 and less than 1.415. Therefore the difference between $\sqrt{2}$ and 1.414 is less than .001. And by the same method closer and closer approximations may be made. See § 339.

Imaginaries will be considered subsequently.

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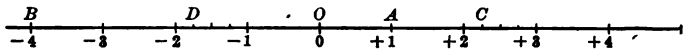
* There are other kinds of irrationals than those here defined, but they will not be considered in this book.

GRAPH OR SCALE OF REAL NUMBERS




**159.** Let $O$ be a fixed point in a straight line; take some line, $u$, $= OA$, for the linear unit; call distances measured from $O$ to the right positive, and distances measured from $O$ to the left negative. Then,

Any point represents the number that expresses its distance and direction from $O$, and is named accordingly.

(1) **Integer points** are the points that represent positive or negative integers; as $A$, $+1$; $B$, $-4$.

(2) **Fraction points** are the points that represent positive or negative fractions; as $C$, $+2\frac{1}{4}$; $D$, $-1\frac{3}{4}$.

(3) **Irrational points** are all the points that are neither integer nor fraction points.

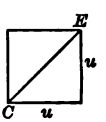
That any irrational is represented by a definite point is shown in advanced algebra.

We may show that $\sqrt{2}$ is a definite point in the scale thus:

The diagonal of a square whose side is $u = 1$, is

$$CE = \sqrt{u^2 + u^2} = \sqrt{1 + 1} = \sqrt{2}.$$

If then we lay off on the scale of numbers $OP = CE$, the point $P$ is $\sqrt{2}$.




**160.** There are, then, three kinds of real numbers, *integers*, *fractions*, and *irrationals;* and to every real number, positive or negative, there corresponds a definite point in a straight line, and to every point in the straight line there corresponds a definite real number. This fact is expressed by saying there is a "**one-to-one correspondence**" between the points of a straight line and real numbers.

Integers and fractions are called **rational numbers.**

**161. Principal roots.** Every number has two square roots. Thus $\sqrt{9}$ is $+3$ or $-3$, for $(+3)^2 = 9$ and $(-3)^2 = 9$. Again, every number has three cube roots, four fourth roots, and, in general, $n$ $n$th roots.

Thus, $\sqrt[3]{8} = 2$, or $-1+\sqrt{-3}$, or $-1-\sqrt{-3}$, since the cube of each of these numbers is 8; and $\sqrt[4]{81} = 3, -3, 3\sqrt{-1}$, or $-3\sqrt{-1}$.

Although the number of roots always equals the index of the root, not more than two of these roots are real; and when there are two real roots, only one of them is positive.

The *principal root* of a number is its real root, if there is but one real root, and its real positive root if there are two real roots.

Thus, considering only the principal root, $\sqrt{36} = 6$, $\sqrt[3]{-125} = -5$, $\sqrt[4]{81} = 3$.

**162.** If two numbers are equal, they may be represented by the same point in the scale of numbers; therefore the following are true because the results correspond to the same point, respectively:

I. *If equal numbers are added to equal numbers, the sums are equal numbers.*

II. *If equal numbers are subtracted from equal numbers, the remainders are equal numbers.*

III. *If equal numbers are multiplied by equal numbers, the products are equal numbers.*

IV. *If equal numbers are divided by equal numbers, the quotients are equal numbers.*

V. *If equal numbers are raised to the same or equal powers, the powers are equal numbers.*

VI. *If the same or equal roots of equal numbers are extracted, the principal or corresponding real roots are equal numbers.*

NOTE. As we shall see further on, there is an exception to axiom IV, when the divisor is zero, § 175.

**163. Finite numbers.** As $x$ increases, becoming 1, 2, 3, ···, the number $10^{3x}$ increases, becoming thousand, million, billion, ···, without end; while the number $\frac{1}{10^{3x}}$ decreases, becoming thousandth, millionth, billionth, ···, without end. Hence, there are numbers (1) which are greater, and (2) others which are smaller, than any given number, however great or small the given number may be. Such numbers are called, respectively, **infinitely large** and **infinitely small** numbers. All other numbers are called **finite numbers.**

An infinitely large number is represented by the symbol $\infty$, and infinitely small numbers are called **infinitesimals.**

### ZERO AND OPERATIONS INVOLVING ZERO

**164. Zero** is the result of subtracting any finite number from itself.

Thus, $1 - 1 = 0$, $a - a = 0$, $b^2 - b^2 = 0$.

**165.** *Zero has a definite numerical value, viz.: none.*

If we ask *how many* birds are in the cage, the answer *none* would be as definite as the answer *four.*

**166.** *The product of zero by any finite number is zero.*

For, let $x$ and $y$ be any finite numbers. Then,

$$0 \times x = (y - y) \times x = xy - xy = 0. \quad \S\S 33, 164.$$

**167.** *The quotient of zero by any finite number is zero.*

For, since $0 = x \times 0, \therefore \frac{0}{x} = 0.$ § 153 (2).

**168.** *The quotient of zero by zero may be any finite number.*

For, since $0 = x \times 0, \therefore \frac{0}{0} = x.$ § 153 (2).

For this reason $\frac{0}{0}$ is called a **symbol of indetermination.**

Thinking only of the intrinsic or abstract value of zero, it may be difficult to understand how one zero can contain another zero an exact number of times; but the difficulty is at once removed by looking to the *definition* of zero for its properties.

Thus, since $1 \times (a - a) = a - a$, therefore $(a - a) \div (a - a) = 1$; hence, in this case $0 \div 0 = 1$.

Again, since $a \times (1 - 1) = a - a$, therefore $(a - a) \div (1 - 1) = a$; hence, in this case $0 \div 0 = a$.

Evidently the quotient of one zero by another depends on the *forms* of the zeros, or the manner in which they are derived from finite numbers.

Thus, a zero may be of the form $1 - 1$, or $a - a$, or $ab - ab$, or $a^3 - a^3$, etc. But even when the forms are known the quotient is often uncertain. For example, to find $(a^2 - a^2) \div (a - a)$, we might say (1) $a^2 - a^2 = a^2(1 - 1)$, $a - a = a(1 - 1)$, and $a^2(1 - 1) \div a(1 - 1) = a$; or (2) $a^2 - a^2 = (a + a)(a - a)$, and $(a + a)(a - a) \div a - a = a + a = 2a$.

It belongs to the differential calculus, and not to elementary algebra, to deduce and establish rules for finding the values of quotients that take the form of $\frac{0}{0}$. However, there is a rule for a simple case that may be given here, which is necessary and sufficient for the purposes of algebra, viz.:

*The quotient of one zero by another is* 1 *if the zeros are the same or corresponding values of the same algebraic number (or variable,* § 310).

Thus, $(x-2) \div (x-2)$, being always equal to 1, whatever may be the value of $x$, is equal to 1 when $x = 2$. Hence, in this case, $\frac{0}{0} = 1$. Again, $\frac{x^2-4}{x-2}$, being always equal to $x+2$, is equal to 4 when $x=2$. That is, in this case, $\frac{0}{0} = 4$.*

Find the value of $\frac{x^2-9}{x-3}$ when $x = 3$.

**169.** *The quotient of a finite number by zero is no finite number.*

For $a$ and $b$ being finite numbers,

let, if possible, $\frac{a}{0} = b$.

Then, $a = 0 \times b$, § 153 (2)

which, by § 166, is impossible.

If $\frac{a}{x} = y$, and we suppose $x$ to become smaller and smaller, $a$ remaining the same, then evidently $y$ will become correspondingly greater and greater. Therefore, when $x$ is infinitely small, $y$ is infinitely great. If, then, we suppose 0 to represent an infinitesimal, we may write

$$\frac{a}{0} = \infty .$$

To distinguish zero as defined in § 164 from zero considered as an infinitesimal, the former is called **absolute zero.**

Therefore, $\frac{a}{0}$, when $a$ is finite and 0 is absolute, has no meaning.

* This principle is embraced in the basal principle of limits, viz.: *If two variables are always equal their limits are equal.*

## EQUALITY CONDITIONS

**170.** A **zero factor** is a factor whose value is zero.

**171. Important properties of zero factors:**

Let $$a \times b \times c = p.$$

If one or more of the factors are zero factors, then $p$ is zero; and if $p$ is zero, then, and only then, are one or more of the factors zero factors.

## EQUATIONS AND EQUALITY CONDITIONS

**172.** DEFINITION. If $x - a = 0, x - b = 0, x - c = 0, \cdots$, then, $$(x-a)(x-b)(x-c)\cdots = 0$$ is called a **conditional equation**, or an **equation** in $x$.

It is so called because it is true on the *conditions* that $x - a = 0$ or $x = a$, $x - b = 0$ or $x = b, \cdots$. The **equality conditions** are the zero factors $x - a$, $x - b$, $x - c, \cdots$.

**173.** PRINCIPLES. For simplicity take the equation

$$(x-a)(x-b)(x-c) = 0. \qquad (1)$$

(1) An equation has as many roots, $a$, $b$, $c$, as it has equality conditions.

(2) Multiplying both members of (1) by $x - d = 0$ gives to (1) a new equality condition, viz.: $x - d = 0$.

(3) Dividing both members of (1) by $x - c$ removes or eliminates the equality condition $x - c = 0$.

(4) Dividing both members of (1) by all the zero factors, $x - c$, $x - b$, $x - a$, eliminates all the equality conditions, and therefore destroys or cancels the equation.

Equation (1) means $(x-a)(x-b)(x-c) = 0 \times 0 \times 0$, in which we may evidently remove $x - c$ from one side if we also remove its equal, 0, from the other side, and this is what we do when we divide (1) by $x - c$ (§ 168).

The preceding principles explain the fallacy in the so-called "proof" that $2 = 1$. Thus:

| | | |
|---|---|---|
| Let | $x = 2.$ | (1) |
| Multiply both sides by $x$, | $x^2 = 2x.$ | (2) |
| Subtract 4 from both sides, | $x^2 - 4 = 2x - 4.$ | (3) |
| Divide both sides by $x - 2$, | $x + 2 = 2.$ | (4) |
| Substitute 2 for $x$ in (4), | $2 + 2 = 2$, or $2 = 1.$ | |

This result is evidently false.

Let us note the false step.

Equation (3) has two equality conditions, $x - 2 = 0$, $x = 0$.

Dividing (3) by $x - 2$ eliminates the condition $x = 2$, and therefore (4) is true only for the other equality condition, $x = 0$. The false step, then, is in making $x = 2$ in (4).

**174.** An **identity** is an equation in which the equality condition is $1 - 1 = 0$.

Thus, the following are identities: $7(1 - 1) = 0$, or $7 = 7$; $(3x - 5)(1 - 1) = 0$, or $3x - 5 = 3x - 5$.

In general, any equation in which the two members are the same, or may be reduced to the same, is an identity. An identity containing letters is true for all values of the letters.

**175.** Dividing an identity by $1 - 1$ destroys the equality.

Thus, since $4 - 4 = 0$, and $2 - 2 = 0$, $4 - 4 = 2 - 2$ is an identity, *regarding the 0's as the same.* Dividing by $1 - 1$, we obtain 4 and 2, and not $4 = 2$, because in the act of division we eliminate the only equality condition, and thereby destroy or cancel the equality.

There is then an exception to the axiom, "If equals are divided by equals the quotients will be equal," viz.: *Provided that the division does not eliminate the only equality condition.*

## INEQUALITIES

**176.** If $a$ and $b$ are positive real numbers, then $a$ is said to be **greater** or **less** than $b$, according as $a - b$ is *positive* or *negative*.

The signs $>$ and $<$ express inequality. Thus, $a > b$ is read, "$a$ is greater than $b$," and $a < b$ is read, "$a$ is less than $b$."

**177.** An **inequality** is a statement in symbols that one of two numbers is greater or less than another.

The **first member** of an inequality is the number to the left of the sign of inequality, and the **second member** is the number to the right of that sign.

Thus, in $3x + 8 > 2y + 7$, $3x + 8$ is the first member and $2y + 7$ is the second member.

**178.** Two inequalities are said to **subsist** in the **same sense** or in the **opposite sense**, according as the signs of the inequalities point in the same or opposite direction.

Thus, $a > b$ and $m > n$, or $a < b$ and $m < n$ subsist in the same sense, while $a > b$ and $m < n$, or $a < b$ and $m > n$ subsist in the opposite sense.

In all that follows, letters are supposed to stand for positive, real, and unequal numbers, unless otherwise stated.

Before memorizing the following principles, the student should illustrate each by one or more numerical examples. He should note carefully wherein the principles of inequalities differ from those of equalities (§ 162).

The student should remember that negative numbers become smaller as their absolute values increase.

Thus, $-1 > -2 > -3 > -4 > -5$.

## PRINCIPLES OF INEQUALITIES

**179.** *An inequality will continue to subsist in the same sense if each member is increased, diminished, multiplied, or divided by the same positive number.*

For if $a > b$, or $a - b$ is positive, then

(1) $a + c - (b + c)$ will be positive, $\therefore a + c > b + c$;

(2) $a - c - (b - c)$ will be positive, $\therefore a - c > b - c$;

(3) $a \times c - b \times c$ will be positive, $\therefore a \times c > b \times c$;

(4) $a \div c - b \div c$ will be positive, $\therefore a \div c > b \div c$.

**180.** *A term can be transposed from one member of an inequality to another provided its sign is changed.*

Let the student show that this principle follows from § 179.

**181.** *An inequality will subsist in the opposite sense if each member is multiplied or divided by the same negative number.*

For, if $a > b$, or $a - b$ is positive, then

(1) $(a - b) \times -c$ will be negative, $\therefore a \times -c < b \times -c$;

(2) $(a - b) \div -c$ will be negative, $\therefore a \div -c < b \div -c$.

**182.** *An inequality will subsist in the opposite sense if the signs of all its terms are changed.*

Let the student show that this principle follows from § 181.

**183.** *An inequality will subsist in the opposite sense if its members are subtracted from equal numbers.*

For, if $a > b$, then evidently $c - a < c - b$.

**184.** *If the first of three numbers is greater than the second, and the second is greater than the third, then the first is greater than the third.*

For, let $a > b$ and $b > c$, then $a - b$ and $b - c$ are positive, and therefore $(a - b) + (b - c)$, or $a - c$ is positive, $\therefore a > c$.

**185.** *The sum or product of the corresponding members of two inequalities that subsist in the same sense is an inequality in the same sense.*

For, let $a>b$ and $c>d$, then, by § 179, $a+c>b+c$ and $b+c>b+d$. Whence, § 184, $a+c>b+d$.

Let the student show, in like manner, that $ac>bd$.

**186.** *The difference or quotient of the corresponding members of two inequalities that subsist in the same sense is indeterminate.*

That is, the result may subsist in the same sense, or in the opposite sense, or it may be an equality.

For example:

| | | | |
|---|---|---|---|
| (1) Subtract: | $12>8$ | $9>8$ | $9>8$ |
| | $3>2$ | $3>2$ | $4>2$ |
| | $9>6$ | $6=6$ | $5<6$ |
| (2) Divide: | $15>8$ | $12>8$ | $9>8$ |
| | $3>2$ | $3>2$ | $3>2$ |
| | $5>4$ | $4=4$ | $3<4$ |

Therefore, *the corresponding members of two inequalities subsisting in the same sense can be subtracted or divided when, and only when, the conditions predetermine the result.*

If $5x+\frac{1}{2}>\frac{2x}{3}+4\frac{5}{6}$, find one limit for $x$.

| | |
|---|---|
| Multiply by 6, | $30x+3>4x+29.$ |
| Transpose, reduce, | $26x>26.$ |
| Divide by 26, | $x>1.$ |

If $2x-3y<2$ (1) and $2x+5y=6$ (2), find one limit for $x$ and one for $y$.

| | |
|---|---|
| (1) − (2), § 179, | $-8y<-4.$ |
| Divide by $-8$, § 181, | $y>\frac{1}{2}.$ |
| From 2, $y=\frac{6-2x}{5}$ | $\therefore \frac{6-2x}{5}>\frac{1}{2},$ |
| whence, | $x<1\frac{3}{4}$, *Ans.* |

Find one limit for $x$:

1. $3x - 5 > 13$.
2. $6 - 5x > 21$.
3. $16 + 3x < 34$.
4. $7x + 8 < 43$.
5. $\frac{4x-2}{3} > \frac{3+5x}{7}$.
6. $\frac{7x}{5} - \frac{5x}{3} > \frac{x}{3} - 3$.

Find the range of the values of $x$:

7. $\begin{cases} 3 - 4x < 7, \\ 2 + x < 4. \end{cases}$
8. $\begin{cases} 2x - 5 > 25, \\ 3x < 2x + 20. \end{cases}$
9. $30 > x + \frac{3x}{2} > 25$.
10. $\begin{cases} 3x + 1 > 13 - x, \\ 4x - 7 < 2x + 3. \end{cases}$

Find one limit for $x$ and one for $y$:

11. $\begin{cases} 2x + 4y > 30, \\ 3x + 2y = 31. \end{cases}$
12. $\begin{cases} 5x + 3y > 121, \\ 7x + 4y = 168. \end{cases}$

Show that $a^2 + b^2 > 2ab$. (A)

| | |
|---|---|
| Evidently, | $(a - b)^2$ is positive. |
| Or, | $(a - b)^2 > 0$. |
| Expand, | $a^2 - 2ab + b^2 > 0$. |
| Transpose, | $a^2 + b^2 > 2ab$. |

Show that the sum of any fraction and its reciprocal is greater than 2.

Divide both sides of (A) by $ab$.

Show that $a^3 + b^3 > a^2b + ab^2$. (B)

| | | |
|---|---|---|
| (A) $\times a$, | $a^3 + ab^2 > 2a^2b$. | (1) |
| (A) $\times b$, | $a^2b + b^3 > 2ab^2$. | (2) |
| [(1) + (2)], | $a^3 + b^3 > a^2b + ab^2$. | |

Show that $a^2 + b^2 + c^2 > ab + ac + bc$. (C)

| | | |
|---|---|---|
| By (A), | $a^2 + b^2 > 2ab$, | (1) |
| | $a^2 + c^2 > 2ac$, | (2) |
| | $b^2 + c^2 > 2bc$. | (3) |
| [(1) + (2) + (3)] $\div 2$, | $a^2 + b^2 + c^2 > ab + ac + bc$. | |

Prove the following:

1. $a^3 + b^3 + c^3 > 3\,abc.$ (D)

HINT. Multiply (C) by $a+b+c$.

2. $\dfrac{r + r'}{2} > \sqrt{rr'}.$ (E)

HINT. In (A) make $a^2 = r$, $b^2 = r'$.

3. $\dfrac{r + r' + r''}{3} > \sqrt[3]{r\,r'\,r''}.$ (F)

4. $\dfrac{a^2 + n}{2\,a} > \sqrt{n}.$ (G)

HINT. In (E) make $r = a$, $r' = \dfrac{n}{a}$.

5. $\dfrac{2\,a^3 + n}{3\,a^2} > \sqrt[3]{n}.$ (H)

HINT. In (F) make $r = r' = a$, $r'' = \dfrac{n}{a^2}$.

6. $\left(\dfrac{a + b}{2}\right)^2 > ab.$ (K)

7. $\dfrac{r+r'+r''+r'''}{4} > \sqrt{rr'r''r'''}.$ (L)

HINT. In (K) make $a = \dfrac{r + r'}{2}$, etc.

8. $\dfrac{a + b}{2} > \dfrac{2\,ab}{a + b}.$ (M)

9. $\sqrt{ab} > \dfrac{2\,ab}{a + b}.$ (N)

10. $\dfrac{a + b}{2} > \sqrt{ab} > \dfrac{2\,ab}{a + b}.$ (O)

11. If $a > \sqrt{n}$, then $a > \dfrac{a^2 + n}{2\,a} > \sqrt{n}.$ (P)

12. If $a > \sqrt[3]{n}$, then $a > \dfrac{2\,a^3 + n}{3\,a^2} > \sqrt[3]{n}.$ (Q)

## PERCENTAGE

**187. Base, Rate, Percentage.** (See arithmetic.)

What is the base? the rate? the percentage?

A dealer bought a number of mules for $2500, and sold them at a gain of 8%. Find his profit.

$8\%$ of $\$2500 = \frac{8}{100}$ of $\$2500 = \$200$, *Ans.*

Here, $2500 is the base, 8% the rate, and $200 the percentage.

Denoting the base by $b$, the rate by $r$, and the percentage by $p$, we have the formula:

$$p = b \times r. \qquad \text{(A)}$$

*The percentage equals the base multiplied by the rate.*

Solve (A) for $b$, and give the interpretation.

Solve (A) for $r$, and give the interpretation.

1. In an army of 24,256 soldiers $6\frac{1}{4}\%$ of the men were killed in battle. How many men were killed?

2. A merchant bought a barrel of flour for $5.25, and sold it at a profit of $.63. What per cent did he gain?

3. $1260 is 84% of what sum?

4. A man sold a horse for $126.60, and gained $5\frac{1}{2}\%$. Find the cost of the horse.

HINT. Let $x$ = the cost, then $x + 5\frac{1}{2}\%$ of $x = 126.50$.

5. How much did a dealer pay for a wagon which he sold for $91.20 at a loss of 5%?

6. The bicycles of James and John cost the same. James sold his for $60, and John his for $64. If James gained 25%, what per cent did John gain?

HINT. Let $x$ = the per cent John gained, then

$$60 \div (1 + 25\%) = 64 \div (1 + x).$$

## INTEREST

**188. Principal, Rate, Interest.** (See arithmetic.)

What is the principal? What is the rate? the interest?

Find the interest of \$240 for 3 yr. 4 mo. 18 da., at 6%.

3 yr. 4 mo. 18 da. = 3 yr. 4.6 mo. = 40.6 mo.
6% per year = $\frac{6}{12}$ or $\frac{1}{2}$% per month.
\$240 × $\frac{1}{2}$% = \$1.20, int. for 1 mo.
40.6 × \$1.20 = \$48.72, int. for 40.6 mo., *Ans.*

Denoting principal by $p$, rate by $r$, time by $t$, and interest by $i$, we have the formula:

$$i = prt. \qquad \text{(B)}$$

When $t$ means number of years, $r$ means rate per year, and when $t$ means number of months, $r$ means rate per month.

Give the interpretation of (B).

Solve (B): for $p$; for $r$; for $t$; and give the interpretation of each.

1. Find the interest of \$1500 for 3 yr. 1 mo. 3 da., at 4%.

2. The interest of \$218 for 3 mo. 18 da. is \$3.27. Find the rate.

3. In what time will \$516.60 yield \$95.0544 interest, at 8%?

4. What principal will produce \$49 interest in 8 yr. 4 mo. at 7%?

5. A man borrowed some money at 6%, kept it 2 yr. 2 mo. 25 da., in which time it amounted to \$762.4322. How much did he borrow?

### THERMOMETERS

**189.** In the Fahrenheit thermometer it is 180° from freezing (32°) to boiling (212°); and in the Centigrade thermometer it is 100° from freezing (0°) to boiling (100°). Hence,

100 Centigrade degrees = 180 Fahrenheit degrees.
∴ 1 Centigrade degree = $\frac{180}{100}$ or $\frac{9}{5}$ Fahrenheit degrees,
and $c$ Centigrade degrees = $\frac{9}{5}$ of $c$ Fahrenheit degrees.

Therefore, if $f$ denotes the number of degrees in any reading of the Fahrenheit thermometer, and $c$ the number of the Centigrade at the same time and place, we have the formula

$$f = \frac{9}{5}c + 32. \qquad (A)$$

1. When the Centigrade registers a temperature of 50°, what is the reading of the Fahrenheit?

2. What is the reading of the Fahrenheit thermometer corresponding to a reading of 80° Centigrade? 20° Centigrade? $-10°$? $-40°$?

3. Solve formula (A) for $c$.

4. When the Fahrenheit registers a temperature of 59°, what is the reading of the Centigrade?

5. What is the registry of the Centigrade corresponding to a reading of 158° Fahrenheit? 239° Fahrenheit? 23°? $-4°$?

6. What does the Centigrade register when the Fahrenheit registers 0°?

7. What is the registry of the Fahrenheit when its reading is twice that of the Centigrade?

8. The normal temperature of the human body by the Fahrenheit is 98.6°. What is it by the Centigrade?

## IX. FACTORS, MULTIPLES, AND FRACTIONS

(Review and Extensions)

### FACTORS

**190.** What is a prime number? What is factoring? Factor:

1. $x^2 - 9\,a^4$.
2. $x^2 + 8\,x + 16$.
3. $x^3 - 8\,y^6$.
4. $x^2 - 10\,x + 25$.
5. $x^3 + 8\,a^3$.
6. $x^2 + 2\,ax + a^2$.
7. $x^4 - 81$.
8. $x^2 - 2\,ax + a^2$.
9. $9\,x^2 - 25$.
10. $x^2 + 7\,x + 12$.
11. $8\,x^3 + 27$.
12. $x^2 - 9\,x + 20$.
13. $8\,x^3 - 125$.
14. $x^2 + 3\,x - 40$.
15. $16\,x^4 - 625$.
16. $x^2 - 2\,x - 63$.

---

17. $cx - nx + cy - ny$.
18. $x^3 + 4\,x^2 + 3\,x + 12$.
19. $x^2 - xy - 3\,x + 3\,y$.
20. $3\,x^3 - 5\,x^2 - 6\,x + 10$.
21. $ab - 3\,bc - 2\,ac + 6\,c^2$.
22. $2\,x^2 - 3\,xy + 4\,ax - 6\,ay$.

---

23. $(x^2 + 1)^2 - (2\,x)^2$.
24. $(x^2 + 8^2) - 16\,x^2$.
25. $(a^2 - 4)^2 - (3\,c)^2$.
26. $(a^2 - 5)^2 - 25\,a^2$.
27. $(c^3 + 3)^2 - (6\,n)^2$.
28. $(x^2 + y^2)^2 - 4\,x^2y^2$.

---

29. $(x^2 + 3\,x + 2)^2 - (2\,x + 1)^2$.
30. $(y^2 - 4\,y + 5)^2 - (3\,y - 2)^2$.

**191. When the trinomial is the difference of two squares.**

1. Factor $a^4 + a^2b^2 + b^4$.

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2). \end{aligned}$$

In this case $a^2b^2$ is both added to and subtracted from the given number, whereby it becomes the difference of two squares, § 93.

2. Factor $x^4 + 2x^2y^2 + 9y^4$.

HINT. Add and subtract $4x^2y^2$.

Factor:

1. $a^4 + a^2 + 1$.
2. $9a^4 + 21a^2c^2 + 25c^4$.
3. $4x^4 + 3x^2 + 1$.
4. $16x^4 - 65x^2y^2 + 49y^4$.
5. $25y^4 + y^2 + 1$.
6. $25x^4 - 41x^2y^2 + 16y^4$.
7. $16a^4 - 17a^2 + 1$.
8. $144 - 217x^2 + 81x^4$.
9. $81 - 34y^2 + y^4$.
10. $49a^4 - 15a^2b^2 + 121b^4$.

**192. When the trinomial is of the form $ax^2 + bx + c$.**

Factor $3x^2 + 13x + 14$.

$$\begin{aligned} 3x^2 + 13x + 14 &= \tfrac{1}{3}(9x^2 + 39x + 42) \\ &= \tfrac{1}{3}[(3x)^2 + 13(3x) + 42] \\ &= \tfrac{1}{3}[(3x + 6)(3x + 7)] \\ &= (x + 2)(3x + 7). \end{aligned}$$

The given number is first multiplied by 3, the coefficient of $x^2$, so as to make the first term a perfect square, and the result multiplied by $\frac{1}{3}$ so as to keep the given number unchanged. $(3x)^2 + 13(3x) + 42$ is factored as if it were $z^2 + 13z + 42$, $z$ taking the place of $3x$.

Factor:

1. $6x^2 + 11x + 3$.
2. $12x^2 - 17x - 40$.
3. $10x^2 - 11x + 3$.
4. $20x^2 - 41x + 20$.
5. $12x^2 + 17x - 7$.
6. $56x^2 - 11x - 15$.
7. $18x^2 - 3x - 10$.
8. $24x^2 - 14x - 5$.
9. $12x^2 + 23x + 10$.
10. $24x^2 + x - 3$.
11. $6x^2 + 13ax + 6a^2$.
12. $24x^2 - 22x + 5$.
13. $8a^2 + 14ab - 15b^2$.
14. $24x^2 + x - 25$.
15. $11a^2 - 23ac + 2c^2$.
16. $12a^4 + a^2b^2 - b^4$.

**193. Finding the value of a polynomial in $x$ for a given value of $x$.**

1. Find the value of $x^3 - 10x^2 + 21x - 7$ for $x = 2$.

FIRST SOLUTION. Substituting 2 for $x$, we have

$$(2)^3 - 10(2)^2 + 21(2) - 7 = 8 - 40 + 42 - 7 = 3, \textit{Ans.}$$

SECOND SOLUTION. By factoring we have

$$x^3 - 10x^2 + 21x - 7 = x[x(x - 10) + 21] - 7.$$

Substitute 2 for $x$: $2[2(2 - 10) + 21] - 7$.

To reduce this begin with the inner parenthesis, and proceed thus:

$2 - 10 = -8$; $2 \times -8 = -16$; $-16 + 21 = +5$; $2 \times 5 = 10$; $10 - 7 = 3$, *Ans.*

The work may be conveniently arranged thus:

The given coefficients: $1 - 10 + 21 - 7$

Multiply the first coefficient 1 by 2, and add the product to $-10$; multiply the sum $(-8)$ by 2 and add the product $(-16)$ to 21; multiply this sum $(+5)$ by 2 and add the product $(+10)$ to $-7$, and this result $(+3)$ is the value required.

$$\begin{array}{r} 2 - 16 + 10 \\ \hline -8 + 5 + 3 \end{array}$$

2. Find the value of $6x^3 - 23x^2 + 16x - 3$ for $x = 3$.

Multiply 6 by 3, add the product to $-23$, and so on. The result is 0.

$$\begin{array}{rrrr} 6 & -23 & +16 & -3 \\ & 18 & -15 & +3 \\ \hline -5 & 1 & 0 & \end{array}$$

When the polynomial is **incomplete**, that is, when one or more of the powers of $x$ are missing, the coefficients of the missing terms are to be supplied with 0's, as in the next example.

3. Find the value of $x^4 - 6x^2 + 3x - 15$ for $x = -3$.

$x^4 - 6x^2 + 3x - 15 = x^4 + 0x^3 - 6x^2 + 3x - 15$

Therefore the coefficients are $1 + 0 - 6 + 3 - 15$,
and the result is $+3$, *Ans.*

$$\begin{array}{r} -3 + 9 - 9 + 18 \\ \hline -3 + 3 - 6 + 3 \end{array}$$

In some of the following the results should be checked by substitution, as in the first of the preceding examples. This will also serve to keep before the student the fact that *the number found is the result of a substitution.*

Find the value of:

1. $x^2 - 8x + 15$ for $x = 3$; $x = 4$; $x = 5$.
2. $x^2 + 3x - 28$ for $x = 4$; $x = 5$; $x = -7$.
3. $6x^2 - 5x + 1$ for $x = \frac{1}{2}$; $x = 2$; $x = \frac{1}{3}$.
4. $x^3 - x + 6$ for $x = 1$; $x = 3$; $x = -2$.
5. $x^3 - 19x + 30$ for $x = 2$; $x = 3$; $x = -5$.
6. $x^3 + 5x^2 - 28$ for $x = 1$; $x = 2$; $x = 3$.
7. $x^4 - 10x^2 + 9$ for $x = 1$; $x = 2$; $x = 3$.
8. $x^3 - 4x^2 - 11x - 30$ for $x = 3$; $x = -2$; $x = -4$.
9. $x^4 - 23x^2 + 18x + 40$ for $x = 2$; $x = 4$; $x = -5$.

**194. The factor theorem.** *One of the factors of a polynomial in $x$ is $x - a$ when, and only when, the result of substituting $a$ for $x$ is zero:*

Thus, $x^3 - 3x^2 + 4x - 12 = 0$ for $x = 3$; therefore $x - 3$ is a factor of $x^3 - 3x^2 + 4x - 12$. Let the student prove this by division.

PROOF. Let $P$ represent any polynomial in $x$. Evidently the division of $P$ by $x - a$ may be continued until the remainder, say $R$, does not contain $x$. Denoting, then, the quotient by $Q$, we have

$$P = Q(x - a) + R. \qquad (1)$$

Now when $a$ is substituted for $x$ in (1), $P$ may or may not become zero, $Q(x - a)$ will become zero, and $R$ will remain unchanged. Hence,

(1) If $P = 0$ for $x = a$, then $0 = 0 + R$. That is $R = 0$ and the division is exact. Hence, $x - a$ is a factor of $P$.

(2) If $P$ does not equal 0 for $x = a$, then $R$ does not equal 0. That is, the division is not exact, and $x - a$ is not a factor of $P$.

In applying the factor theorem the trial divisor must always be expressed under the form of $x - a$.

1. Find a factor of $x^3 + 3x^2 - 9x - 11$.

If there is a factor like $x - a$, the only possible values of $a$ are the divisors of 11, viz.: 1, $-1$, 11, $-11$.

To test the factor $x + 1$, we write it under the form $x - (-1)$. Now by the factor theorem the *remainder* is the result of substituting $-1$ for $x$ in $x^3 + 3x^2 - 9x - 11$.

By § 193,

$$\begin{array}{r} 1 + 3 - 9 - 11 \\ -1 - 2 + 11 \\ \hline 2 - 11 \quad 0 \end{array}$$

The result (or remainder) being 0, $x + 1$ is a factor.

2. Find the remainder when $x^3 - 9x^2 + 23x - 15$ is divided by $x - 5$.

$$\begin{array}{r} 1 - 9 + 23 - 15 \\ \quad 5 - 20 + 15 \\ \hline -4 + 3 \quad 0. \end{array}$$ *Ans.*

Hence, $x - 5$ is a factor of the given polynomial.

3. In like manner find which of $x - 1$, $x + 1$, $x - 2$, $x + 2$, $x + 3$ are factors of $x^3 - 6x^2 + 11x - 6$.

4. Find the factors of $x^3 - 2x^2 - 4x + 3$.

The exact divisors of 3 are 1, $-1$, 3, $-3$.

The result of substituting 3 for $x$ in the given polynomial is 0, therefore one of the factors is $x - 3$. Dividing the given polynomial by $x - 3$ we obtain the other factor $x^2 + x - 1$.

As neither $+1$ nor $-1$, the only divisors of $-1$, will reduce $x^2 + x - 1$ to zero, this number cannot be resolved into rational factors.

Factor:

1. $x^3 - 7x + 6$.
2. $x^3 - 5x^2 + 8x - 4$.
3. $x^3 + 5x + 6$.
4. $x^3 + 14x^2 + 35x + 22$.
5. $x^3 - 3x + 2$.
6. $x^3 - 11x^2 + 31x - 21$.
7. $x^3 + 3x^2 - 4$.
8. $x^3 - 2x^2 - 23x + 60$.
9. $x^3 - 13x + 12$.
10. $x^3 - 10x^2 + 33x - 36$.
11. $x^3 + 2x - 33$.
12. $3x^3 - 2x^2 + 5x - 6$.
13. $x^3 + x^2 - 150$.
14. $2x^3 + 3x^2 - 3x - 4$.
15. $a^3 - 3a - 52$.
16. $3c^3 + 5c^2 - 7c - 1$.
17. $b^4 + 3b - 22$.
18. $s^4 - 11s^2 - 18s - 8$.
19. $r^4 - 10r^2 + 9$.
20. $n^4 - n^3 - 7n^2 + n + 6$.

Show by the factor theorem that :

1. $x - a$ is a factor of $x^n - a^n$, if $n$ is any integer.
2. $x + a$ is a factor of $x^n - a^n$, if $n$ is any even integer.
3. $x + a$ is a factor of $x^n + a^n$, if $n$ is any odd integer.
4. Show by the same theorem that neither $x + a$ nor $x - a$ is a factor of $x^n + a^n$ if $n$ is any even integer.

Factor :

5. $x^5 - a^5$.
6. $x^4 - y^4$.
7. $c^5 + 32$.
8. $x^6 - a^6$.
9. $x^7 + y^7$.
10. $r^4 - 81$.

### COMMON DIVISORS OR FACTORS

**195** What is a common factor of two or more numbers?

What is the highest common factor of two or more numbers?

State the principle, § 101.

Find the H.C.F. of :

1. $x^2 - 16$ and $x^2 + 9x + 20$.
2. $x^2 - 25$ and $x^2 - 9x + 20$.
3. $x^2 - 81$ and $x^2 + 3x - 54$.
4. $x^2 - 7x - 30$ and $x^2 + 12x + 27$.
5. $x^2 + 6x - 16$ and $x^2 + 11x + 24$.
6. $x^2 - 5x - 84$ and $x^2 - 6x - 91$.
7. $x^2 - 7x + 10$ and $x^3 - 6x^2 + 11x - 6$.

$x^2 - 7x + 10 = (x - 2)(x - 5)$. Now by trial we find that $x - 2$ will divide $x^3 - 6x^2 + 11x - 6$, and $x - 5$ will not. Hence, $x - 2$ is the G. C. D.

8. $x^2 + x - 6$ and $2x^3 - 12x^2 + 21x - 10$.
9. $a^2 - 7a + 12$ and $5a^3 - 18a^2 + 11a - 6$.

**196. Finding the H. C. F. of polynomials neither of which is easily factored.**

The process is essentially the same as that used in arithmetic to find the G. C. D. of two integers.

Thus, to find the G. C. D. of 1517 and 533, divide the greater integer by the less; then divide the less integer (533) by the remainder; and continue to divide the last divisor by the last remainder until nothing remains; then the last divisor (41) is the G. C. D. required.

$$
\begin{array}{r}
533)1517(2 \\
\underline{1066} \\
451)533(1 \\
\underline{451} \\
82)451(5 \\
\underline{410} \\
41)82(2 \\
\underline{82}
\end{array}
$$

Now, similarly, to find the H. C. F. of $x^3 + 4x^2 + 7x + 6$ and $x^2 + 3x + 2$, we proceed as follows:

$$
\begin{array}{r}
x^2 + 3x + 2)x^3 + 4x^2 + 7x + 6(x + 1 \\
\underline{x^3 + 3x^2 + 2x} \\
x^2 + 5x + 6 \\
\underline{x^2 + 3x + 2} \\
2x + 4
\end{array}
$$

The first remainder is $2x + 4$, which we divide by 2 and reject the 2, since it is not a factor of the given numbers.

The H. C. F. is $x + 2$.

$$
\begin{array}{r}
x + 2)x^2 + 3x + 2(x + 1 \\
\underline{x^2 + 2x} \\
x + 2 \\
\underline{x + 2}
\end{array}
$$

The work may be conveniently arranged thus:

| | | |
|---|---|---|
| $x^3 + 4x^2 + 7x + 6$ | Quotients | $x^2 + 3x + 2$ |
| $\underline{x^3 + 3x^2 + 2x}$ | $x + 1$ | $\underline{x^2 + 2x}$ |
| $x^2 + 5x + 6$ | | $x + 2$ |
| $\underline{x^2 + 3x + 2}$ | $x + 1$ | $x + 2$ |
| $2x + 4$ | | |
| $x + 2$ | | |

The preceding process is based on the following considerations:

Let $A$ and $B$ represent two polynomials in $x$, and suppose the quotient of $A$ by $B$ to be $Q$ and the remainder $R$.

Then, $$A = QB + R.$$

This equation shows that every divisor of $B$ and $R$ is a divisor of $A$ also, and every divisor of $A$ and $B$ is a divisor of $R$ also, otherwise we should have an integer equal to an irreducible fraction. Therefore, the H. C. F. of $A$ and $B$ is the same as that of $B$ and $R$.

Again, if the quotient of $B$ by $R$ is $Q'$ and the remainder $R'$, then, for the same reason, the H. C. F. of $B$ and $R$ is the same as that of $R$ and $R'$; and so on. That is, the H. C. F. of $A$ and $B$ is the same as that of any two *successive remainders*.

Therefore, if the final remainder is zero, then the H. C. F. of the two preceding remainders is the last divisor itself; hence, the last divisor is the H. C. F. of $A$ and $B$.

The H. C. F. of two algebraic numbers is evidently not altered by multiplying or dividing either of them by any number that is not a factor of the other. This fact is useful in avoiding fractional coefficients and otherwise facilitating the process of finding the H. C. F.

Thus, the H. C. F. of $3x^4 - 12x^3 + 9x^2 - 15x$ and $5x^3 - 15x^2 + 30$ is the same as that of $x^3 - 4x^2 + 3x - 5$ and $x^3 - 3x^2 + 6$, which are, respectively, the quotients of the given numbers by $3x$ and 5.

A convenient check on the accuracy of the work is to substitute some number, as 1 or 10, for $x$ in both the given numbers and also in the H. C. F., and find by division whether the latter is a divisor of both the former results.

All the principles involved in finding the H. C. F. are illustrated in the solution of the next example.

Find the H. C. F. of $4x^5 + 9x^4 + 2x^3 - 2x^2 - 4x$ and $6x^4 + 10x^3 - 2x^2 + 4x$.

Since the factor $x$ is common to both, we divide both by $x$, and retain the $x$ as a factor of the H. C. F. Again, since 2 is a factor of the second number and not of the first, we divide the second by 2 and throw it away.

The two numbers thus become

$$4x^4 + 9x^3 + 2x^2 - 2x - 4 \text{ and } 3x^3 + 5x^2 - x + 2.$$

Now, in order that the first may be divisible by the second, we multiply it by 3, and then proceed thus:

$$\begin{array}{r|l} 12x^4 + 27x^3 + 6x^2 - 6x - 12 & 3x^3 + 5x^2 - x + 2. \\ \underline{12x^4 + 20x^3 - 4x^2 + 8x} & 4x + 7. \\ 7x^3 + 10x^2 - 14x - 12 & \end{array}$$

We multiply this remainder by 3 to make it divisible by the divisor and continue thus:

$$\begin{array}{r} 21x^3 + 30x^2 - 42x - 36 \\ \underline{21x^3 + 35x^2 - 7x + 14} \\ -5x^2 - 35x - 50 \\ \text{or } -5(x^2 + 7x + 10) \end{array}$$

Now rejecting the factor $-5$, we continue thus:

$$\begin{array}{r|l} 3x^3 + 5x^2 - x + 2 & x^2 + 7x + 10 \\ \underline{3x^3 + 21x^2 + 30x} & 3x - 16 \\ -16x^2 - 31x + 2 & \\ \underline{-16x^2 - 112x - 160} & \\ +81x + 162 & \\ \text{or } 81(x + 2) & \end{array}$$

Rejecting the factor 81, we proceed thus:

$$\begin{array}{r|l} x^2 + 7x + 10 & x + 2 \\ \underline{x^2 + 2x} & x + 5 \\ 5x + 10 & \\ \underline{5x + 10} & \end{array}$$

Hence, $x(x + 2)$ or $x^2 + 2x$ is the H. C. F.

Find the H. C. F. of:

1. $x^2 + 20\,x + 91$ and $x^2 - 10\,x - 119$.
2. $x^2 + 28\,x + 187$ and $x^2 - 2\,x - 143$.
3. $3\,x^2 - 6\,x + 3$ and $6\,x^2 + 6\,x - 12$.
4. $10\,x^2 + x - 60$ and $6\,x^2 + x - 35$.
5. $28\,x^2 - 19\,x + 3$ and $20\,x^2 + 7\,x - 3$.
6. $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ and $a^3 + b^3$.
7. $3\,x^3 + 6\,x^2 - 24\,x$ and $6\,x^3 - 96\,x$.
8. $x^2 - a^2 + 2\,xy + y^2$ and $x^2 + a^2 + 2\,ax - y^2$.
9. $y^3 - 5\,y^2z - 4\,yz^2 + 2\,z^3$ and $7\,y^2 + 10\,yz + 3\,z^2$.
10. $7\,a^2 - 23\,ab + 6\,b^2$ and $5\,a^3 - 18\,a^2b + 11\,ab^2 - 6\,b^3$.
11. $x^4 - 8\,x^3 + 21\,x^2 - 20\,x + 4$ and $2\,x^3 - 12\,x^2 + 21\,x - 10$.
12. $2\,x^4 - 12\,x^3 + 19\,x^2 - 6\,x + 9$ and $4\,x^3 - 18\,x^2 + 19\,x - 3$.
13. $6\,x^4 + x^3 - x$ and $4\,x^3 - 6\,x^2 - 4\,x + 3$.
14. $2\,x^4 + 11\,x^3 - 13\,x^2 - 99\,x - 45$ and $2\,x^3 - 7\,x^2 - 46\,x - 21$.
15. $3\,x^5 - 10\,x^3 + 15\,x + 8$ and $x^5 - 2\,x^4 - 6\,x^3 + 4\,x^2 + 13\,x + 6$.

## COMMON MULTIPLES

**197.** What is a common multiple of two or more numbers?

What is the lowest common multiple of two or more numbers?

State the principle, § 107.

**198.** *The lowest common multiple of two numbers may be found by dividing either of them by their highest common factor and multiplying the quotient by the other.*

For, let $AD$ and $BD$ be the numbers, $D$ being their H. C. F. Then, evidently, their L. C. M. is $ABD$, which may be found thus: First find $D$; then divide $AD$ by $D$, and multiply the quotient ($A$) by $BD$.

Find the L. C. M. of $x^3-6x^2+11x-6$ and $x^3-9x^2+26x-24$.

By factoring, or by § 196 we find the H. C. F. to be $x^2-5x+6$.

$$(x^3-6x^2+11x-6)\div(x^2-5x+6)=x-1.$$

$$(x^3-9x^2+26x-24)(x-1)=(x-1)(x-4)(x-2)(x-3). \quad \textit{Ans.}$$

When possible, it is always best to express the L. C. M. in terms of its factors.

Find the L. C. M. of:

1. $x^3+6x^2+11x+6$ and $x^3+7x^2+14x+8$.
2. $x^3+4x^2-4x-16$ and $x^3-4x^2-4x+16$.
3. $x^3-5x^2-4x+20$ and $x^3+2x^2-25x-50$.
4. $3x^4-x^3-5x^2+x+2$ and $2x^3-3x^2-2x+3$.

**199.** It follows from § 108 that if $A$, $B$, and $C$ stand for any three given numbers, and if $G$ is the L. C. M. of $A$ and $B$, while $H$ is the L. C. M. of $G$ and $C$, then $H$ is the L. C. M. of $A$, $B$, and $C$.

5. Find the L. C. M. of $x^2-4x+3$, $x^2-7x+12$, $x^2-11x+28$.

The L. C. M. $x^2-3x+2$ and $x^2-2x+12$ is $(x-1)(x-3)(x-4)$ and the L. C. M. of $(x-1)(x-3)(x-4)$ and $x^2-11x+28$ is $(x-1)(x-3)(x-4)(x-7)$. *Ans.*

Find the L. C. M. of:

6. $x^2-9$, $x^2+2x-15$, and $x^2-4x-21$.
7. $x^2+x-30$, $x^2+12x+36$, and $x^2-10x+25$.
8. $y^2+y-6$, $y^3-2y^2-y+2$, and $y^3+3y^2-6y-8$.
9. $6z^2+7z-3$, $3z^2+11z-4$, $2z^2+11z+12$.
10. $6c^2+39c+45$, $3c^2+14c-5$, $6c^3+7c^2-3c$.
11. $a^2+2ab+b^2$, $a^2-b^2$, $a^3+2a^2b+2ab^2+b^3$.
12. $y^4+y^3+2y^2-y-3$, $y^4+2y^3+2y^2-y-2$.

## FRACTIONS

**200.** Reduce to lowest terms:

1. $\frac{x^2+5x-36}{x^2+2x-24}$.
2. $\frac{2x^2-xy-15y^2}{2x^2-3xy-9y^2}$.
3. $\frac{x^2+x-72}{x^2-x-90}$.
4. $\frac{x^3-x^2-x-2}{x^2-6x+8}$.
5. $\frac{a^2-2a-3}{a^2-10a+21}$.
6. $\frac{a^3-3a^2+4a-2}{a^3-a^2-2a+2}$.
7. $\frac{8x^2-6x-35}{12x^2-23x-77}$.
8. $\frac{6z^3-23z^2+16z-3}{6z^3-17z^2+11z-2}$.
9. $\frac{a^2-(b-c)^2}{(a+b)^2-c^2}$.
10. $\frac{4y^3+8y^2-3y+5}{6y^3-5y^2+4y-1}$.
11. $\frac{a^4+a^3+a^2b^2+b^3+b^4}{a^2+a+ab+b+b^2}$.
12. $\frac{27x^5+8x^2-3x+1}{9x^4-x^2+2x-1}$.

**201.** Change to integral or mixed numbers:

1. $\frac{x^2+5x+6}{x+3}$.
2. $\frac{12x^2-4xy+10y^2}{6x-20y}$.
3. $\frac{5x^2+3x+1}{x+4}$.
4. $\frac{a^4-2a^2b^2+b^4}{a^2-2ab+b^2}$.
5. $\frac{7a^2-5a+7}{a+6}$.
6. $\frac{x^2-11x+24}{x^2+x-12}$.
7. $\frac{a^3+b^3}{a+b}$.
8. $\frac{x^3+2x^2-63x}{x^2-2x-35}$.
9. $\frac{a^4+b^4}{a+b}$.
10. $\frac{12x^3+4x^2-27x-9}{2x^2+5x+3}$.

**202.** Reduce to fractional forms:

1. $2x+1+\frac{3x+1}{x}$.
2. $x+2+\frac{(x-2)^2}{x+2}$.
3. $2x+1-\frac{3x+1}{x}$.
4. $x+3-\frac{(x-3)^2}{x+3}$.
5. $x^2+3x+2+\frac{x^3-3x^2+2x+4}{x-2}$.
6. $a^2-2a+3-\frac{a^3-3a^2+4a-3}{a-1}$.

**203.** Reduce to least common denominator:

1. $\frac{x^2}{x^2-9}$, $\frac{x+2}{x^2+3x}$, $\frac{x+3}{x^2-3x}$.
2. $\frac{9}{8(1+x)}$, $\frac{3x}{4(1+x)^2}$, $\frac{x^2}{2(1+x)^3}$.
3. $\frac{1}{x^2-5x+6}$, $\frac{1}{x^2-7x+12}$, $\frac{1}{x^2-6x+8}$.
4. $\frac{x+1}{x^2-x-6}$, $\frac{x-1}{x^2-3x-10}$, $\frac{x}{x^2-8x+15}$.
5. $\frac{1}{a^2+ab+b^2}$, $\frac{1}{a^2-ab+b^2}$, $\frac{1}{a^6-b^6}$.

**204. Errors often made by students in fractions.** The sign $\neq$ means, and is read, "is not equal to."

Students sometimes cancel common addends in the terms of a fraction, and confound *reducing fractions to a common denominator* with *clearing an equation of fractions*.

Thus, $\frac{a^2+8a+15}{a^2+9a+18}\neq\frac{a^2+8a+15}{a^2+9a+18}$ or $\frac{8a+15}{9a+18}$, but $=\frac{a+5}{a+6}$. § 111.

Again, $\frac{3}{a+3}+\frac{2}{a+1}\neq 5a+9$, but $\frac{5a+9}{a^2+4a+3}$.

The best method of correcting the misconceptions from which these mistakes arise is to check the results of addition and multiplication of fractions by substituting numbers for the letters, taking care not to substitute such a number as will make any denominator zero. § 169.

Thus, substituting 4 for $a$ in (1) and (2), we have (1) $\frac{8}{7}\frac{8}{8} \neq \frac{4}{4}\frac{7}{4}$, but $\frac{8}{10}$; (2) $\frac{8}{7}+\frac{8}{8} \neq 29$, but $\frac{2}{5}\frac{9}{5}$, because $\frac{8}{7}=\frac{1}{3}\frac{5}{5}$, $\frac{7}{5}=\frac{1}{3}\frac{4}{5}$, and $\frac{1}{3}\frac{5}{5}+\frac{1}{3}\frac{4}{5}=\frac{2}{5}\frac{9}{5}$.

Perform the operations indicated in the following, and check the results for one or more values of $x$:

1. $\dfrac{6}{3x-1}+\dfrac{5}{3x+1}-\dfrac{15x+1}{9x^2-1}$.

2. $\dfrac{4}{x+1}-\dfrac{2}{(x+1)^2}+\dfrac{1}{(x+1)^3}$.

3. $\dfrac{x+2}{(x+3)(x+4)}+\dfrac{x-3}{(x+4)(x+5)}$.

4. $\dfrac{x-6}{(x+4)(x-5)}-\dfrac{x+7}{(x-5)(x+6)}$.

5. $\dfrac{x+3}{(x+1)(x+2)}-\dfrac{x+1}{(x+2)(x+3)}+\dfrac{x+2}{(x+1)(x+3)}$.

6. $\dfrac{x+4}{x^2+5x+6}+\dfrac{x+3}{x^2+6x+8}-\dfrac{x+2}{x^2+7x+12}$.

7. $\dfrac{x+6}{x^2-9x+20}-\dfrac{2(x+5)}{x^2-10x+24}+\dfrac{x+4}{x^2-11x+30}$.

Simplify:

8. $\dfrac{3}{1-2x}-\dfrac{7}{1+2x}-\dfrac{4-20x}{4x^2-1}$.

9. $\dfrac{5}{2(x+1)}-\dfrac{1}{10(x-1)}-\dfrac{24}{5(2x+3)}$.

10. $\dfrac{(b-a)}{x-b}-\dfrac{a-2b}{x+b}+\dfrac{3x(a-b)}{x^2-b^2}$.

11. $\dfrac{3a-4b}{7}-\dfrac{2a-b-c}{3}+\dfrac{15a-4c}{12}-\dfrac{a-4b}{21}$.

12. $\dfrac{1}{a+b}+\dfrac{b}{a^2-b^2}-\dfrac{a}{a^2+b^2}$.

13. $\dfrac{a}{a-x}+\dfrac{3a}{a+x}-\dfrac{2ax}{a^2-x^2}$.

14. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}$.

15. $\dfrac{bc}{(c-a)(a-b)}+\dfrac{ca}{(a-b)(b-c)}+\dfrac{ab}{(b-c)(c-a)}$.

16. $\dfrac{1}{(b-c)(c-a)}+\dfrac{1}{(c-a)(a-b)}+\dfrac{1}{(a-b)(b-c)}$.

17. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ac}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}$.

18. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ac}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}$.

**205.** Multiply $\frac{x^2-1}{x^2-4}$ by $\frac{x^2-x-6}{x^2+2x+1}$.

Factor the terms and cancel common factors.

$$\frac{x^2-1}{x^2-4} \times \frac{x^2-x-6}{x^2+2x+1} = \frac{(x-1)(x+1)(x+2)(x-3)}{(x-2)(x+2)(x+1)(x+1)} = \frac{x^2-4x+3}{x^2-x-2}.$$

Perform the following indicated operations:

1. $\frac{x^2+5x+6}{x^2+7x+10} \times \frac{x^2+9x+20}{x^2-3x-28}$.

2. $\frac{x^2-4x+3}{x^2+4x+3} \times \frac{x^2+5x+6}{x^2-7x+12}$.

3. $\frac{x^2-9}{x^2-16} \times \frac{(x+4)^2}{(x-3)^2} \times \frac{x^2-7x+12}{x^2+7x+12}$.

4. $\frac{x^2+8x+15}{x^2+5x-14} \times \frac{x^2-7x+10}{x^2-x-12} \times \frac{x^2+3x-28}{x^2-25}$.

5. $\frac{3ax}{4by} \times \frac{a^2-x^2}{c^2-x^2} \times \frac{bc+bx}{a^2+ax} \times \frac{c-x}{a-x}$.

6. $\frac{1-x^2}{1+y} \times \frac{1-y^2}{x+x^2} \times \left(1+\frac{x}{1-x}\right)$.

7. $\frac{a^4-b^4}{(a-b)^2} \times \frac{a-b}{a^2+ab} \times \frac{ab}{a^2+b^2}$.

8. $\left(\frac{x^2}{a^2}-\frac{x}{a}+1\right) \times \left(\frac{x^2}{a^2}+\frac{x}{a}+1\right)$.

9. $(x^2-x+1) \times \left(1+\frac{1}{x}+\frac{1}{x^2}\right)$.

10. $\left(\frac{a^3-b^3}{a^3+b^3}\right)\left(\frac{a+b}{a-b}\right)\left(\frac{a^2-ab+b^2}{a^2+ab+b^2}\right)$.

**206.** 1. Divide $\frac{x^2+x-2}{x^2-7x}$ by $\frac{x^2+2x}{x^2-13x+42}$.

2. Divide $\frac{a^2}{b^2}-\frac{b^2}{a^2}$ by $\frac{a}{b}+\frac{b}{a}$.

3. Divide $\frac{x^2+xy+y^2}{x^2-xy+y^2}$ by $\frac{x+y}{x-y}$.

Perform the following indicated operations:

4. $\left(x^2-\frac{1}{x^2}\right)\div\left(x+\frac{1}{x}\right)$.

5. $\left(x^3-\frac{1}{x^3}\right)\div\left(x-\frac{1}{x}\right)$.

6. $\left(x^2+2+\frac{1}{x^2}\right)\div\left(x+\frac{1}{x}\right)$.

7. $\left(x^2+1+\frac{1}{x^2}\right)\div\left(x-1+\frac{1}{x}\right)$.

8. $(a^2-b^2-c^2+2bc)\div\frac{a+b-c}{a+b+c}$.

9. $(a^2-b^2-c^2-2bc)\div\frac{a+b+c}{a+b-c}$.

10. $\frac{x^2-14x+49}{x^2-12x+36}\div\frac{x^2-15x+56}{x^2-11x+30}$.

11. $\frac{a^2-7a+6}{a^2+3a-4}\div\frac{a^2-14a+48}{a^2+10a+24}\div\frac{a^2+6a}{a^3-8a^2}$.

12. $\frac{(x-a)^2-b^2}{(x-b)^2-a^2}\div\frac{x^2-(a-b)^2}{x^2-(b-a)^2}$.

13. $\frac{(x+a)^2-(b+c)^2}{(x+b)^2-(a+c)^2}\div\frac{(x-b)^2-(c-a)^2}{(x-a)^2-(c-b)^2}$.

**207.** Simplify:

1. $\dfrac{\frac{3}{4}+\frac{3}{5}}{\frac{2}{3}-\frac{1}{2}}$.

2. $\dfrac{x-1+\frac{1}{x+1}}{x+1+\frac{1}{x-1}}$.

3. $\dfrac{\frac{2x}{5}+\frac{3x}{8}}{3x+\frac{7x}{10}}$.

4. $\dfrac{3x+\frac{27x}{2x+3}}{2x+\frac{18x}{2x-3}}$.

5. $\dfrac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}$.

6. $\dfrac{\frac{x+1}{x-1}+\frac{x-1}{x+1}}{\frac{x+1}{x-1}-\frac{x-1}{x+1}}$.

7. $\dfrac{\frac{x}{x+y}+\frac{y}{x-y}}{\frac{x}{x-y}-\frac{y}{x+y}}$.

8. $\dfrac{\frac{x^2-17x+72}{x^2+22x+120}}{\frac{x^2-21x+108}{x^2+18x+80}}$.

9. $\dfrac{\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}}{\frac{x+y}{x-y}-\frac{x-y}{x+y}}$.

10. $\dfrac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}}$.

11. $\dfrac{1}{x+\frac{1}{x+\frac{1}{x}}}$.

12. $\dfrac{1}{\frac{1}{b+\frac{1}{c}}}$.

13. $\dfrac{a}{b+\frac{c}{d+\frac{e}{f}}}$.

14. $\dfrac{1}{x+\frac{1}{1+\frac{x+1}{3-x}}}$.

### AVERAGES

**208.** As the student continues his studies and observations he will come to see that the idea of an **average** of several quantities of the same kind is as valuable and as frequently employed as any within the whole range of science or commerce.

It gives one a single representative magnitude, easily grasped and remembered, of an indefinite number of perhaps indefinitely apprehended objects. For instance, a clear conception of the duration of human life is obtained by taking the average; an idea of the climate of a place is formed by learning the average of the temperatures of each day for a year; and the average citizen of a country furnishes a definite idea of the morals and education of its people. *The chief value of all statistics depends on the doctrine of averages.*

**209.** Primarily, the **average** or **mean value** of several quantities of the same kind is the quotient of their sum by their number.

FORMULA. If $a$, $b$, $c$, $\cdots$ are a series of values whose number is $n$, and whose average or mean value is $m$, then,

$$m = \frac{a + b + c + \cdots}{n}.$$

1. The ages of three girls are, respectively, 11, 12, and 14 years. Find the average of their ages.

2. The weights of a football eleven are, respectively, 162, 165, 170, 171, 180, 182, 187, 193, 198, 205, 213 pounds. Find the average weight.

**3.** The grades of a student are 92% in arithmetic, 95% in history, 89% in geography, and 87% in English. What is the average?

Instead of adding the grades and dividing by 4, it will be easier to add to 90 the average of the excesses of the grades over 90. Thus, $92 = 90 + 2$, $95 = 90 + 5$, $89 = 90 - 1$, $87 = 90 - 3$; $\frac{1}{4}(2 + 5 - 1 - 3) = \frac{3}{4}$; $90 + \frac{3}{4} = 90\frac{3}{4}$. *Ans.*

**4.** A man measured an iron rail six times, getting 29 ft. $11\frac{3}{4}$ in. *one* time, 29 ft. $11\frac{5}{16}$ in. *two* times, and 29 ft. $11\frac{5}{8}$ in. *three* times. Find the most probable length of the rail.

In problems like this the numbers *one*, *two*, *three* are called the **weights** of the corresponding measurements or values. Each value is multiplied by its weight, and the sum of the products divided by the number of weights (6).

Or, to 29 ft. 11 in. add $\frac{1}{8}$ of the excesses; that is, $\frac{1}{8}$ of $(1 \times \frac{3}{4} + 2 \times \frac{5}{16} + 3 \times \frac{5}{8}) = \frac{1}{8}$ of $\frac{31}{8} = \frac{31}{64}$, giving 29 ft. $11\frac{31}{64}$ in.

**5.** George recited 5 times a week in English, 3 times a week in algebra, and 2 times a week in drawing, and his average grade in English was 84%, in algebra 70%, and in drawing 75%. Find his average grade, allowing the number of hours a study is taken a week to be its weight.

**6.** Ella's grade in English for the first term is 87%. How much must it be for the second term so that the grade for the year shall be 91%?

**7.** In a certain college the pass mark is 75%. If Walter's standing for the first seven months is 70%, 78%, 80%, 76%, 68%, 72%, and 66%, how much must it be for the eighth month in order that he may barely pass?

**8.** B is 6 yr. older than A, and C is 9 yr. older than

B, and the average of their ages is 39 yr. Find the age of each.

9. A grocer has two kinds of sugars, worth 4 and $6\frac{1}{4}$¢ a pound. How many pounds of the first must be mixed with 8 lb. of the second so that the average value of the mixtures shall be 5¢ a pound?

10. A class in grammar is divided into three sections. The average age of the pupils in the class is $13\frac{1}{2}$ yr., and of the three sections 12, 14, and 16 yr., respectively. If there are 15 pupils in the first section and 10 in the second, how many are there in the third section?

11. Henry rode from A to B in 4 hr. at the rate of 6 mi. per hour, from B to C at the rate of 8 mi. per hour, and his average rate from A to C was $7\frac{1}{5}$ mi. per hour. How many hours was he in going from B to C?

12. How much milk testing 4 % fat must be mixed with 72 gal. of cream testing 30 % fat so as to produce an average cream testing 16 % fat?

HINT. Let $x$ = the number of gallons. Then, $\dfrac{4\%\text{ of }x + 30\%\text{ of }72}{x + 72} = 16\%$.

13. A dairyman contracts to deliver cream testing $16\frac{2}{3}$ % fat. Upon testing, he finds that it tests 25 % fat. How much skim milk (testing 0 % fat) must be mixed with 100 lb. of the cream to reduce it to the average agreed upon?

14. A dairyman contracts to deliver 100 lb. of cream testing 21 % fat. Now he has a cream testing 28 % fat and another testing 18 % fat. How much of each must he put into a mixture so as to have the required amount and quality?

## X. EQUATIONS OF THE FIRST DEGREE

(Review and Extensions)

### EQUATIONS OF TWO UNKNOWN NUMBERS

**210. Eliminating by Substitution.**

Solve $\begin{cases} 5x + 4y = 7, & (1) \\ 2x - 3y = 12. & (2) \end{cases}$

These equations were solved on page 41 by addition and subtraction, and on page 110 by comparison. We will now solve them by another method, called **elimination by substitution.**

Solve (1) for $x$, $\quad x = \dfrac{7 - 4y}{5}$. (3)

*Substitute* this value of $x$ for $x$ in (2),

$$2\left(\frac{7 - 4y}{5}\right) - 3y = 12. \quad (4)$$

Solve (4) for $y$, $\quad y = -2$.

Substitute $-2$ for $y$ in (3), $\quad x = 3$.

In like manner solve:

**1.** $\begin{cases} 5x + 6y = 49, \\ 7x - 4y = 19. \end{cases}$

**2.** $\begin{cases} 3x - 2y = 1, \\ 3y - 4x = 1. \end{cases}$

**3.** $\begin{cases} \dfrac{x}{5} + \dfrac{y}{6} = 18, \\ \dfrac{x}{2} - \dfrac{y}{4} = 21. \end{cases}$

**4.** $\begin{cases} \dfrac{x}{3} + \dfrac{y}{4} = 9, \\ \dfrac{x}{4} + \dfrac{y}{5} = 7. \end{cases}$

**5.** $\begin{cases} 3x - 5y = 13, \\ 2x + 7y = 81. \end{cases}$

**6.** $\begin{cases} 11x + 3y = 100, \\ 4x - 7y = 4. \end{cases}$

**211.** The following may be solved by addition and subtraction, comparison, or substitution, as the teacher may think best:

**1.** $\begin{cases} 5x+7y=43, \\ 11x+9y=69. \end{cases}$

**2.** $\begin{cases} 7.7y-3x=29.5, \\ 2.1y+2x=16.5. \end{cases}$

**3.** $\begin{cases} \frac{x}{8}+\frac{y}{9}=42, \\ \frac{x}{9}+\frac{y}{8}=43. \end{cases}$

**4.** $\begin{cases} \frac{x+y}{2}-\frac{x-y}{3}=8. \\ \frac{x+y}{3}+\frac{x-y}{4}=11. \end{cases}$

**5.** $\begin{cases} x+y=s, \\ x-y=d. \end{cases}$

**6.** $\begin{cases} 3x+2y=a, \\ 4x-3y=b. \end{cases}$

**7.** $\begin{cases} ax+by=m, & (1) \\ cx+dy=n. & (2) \end{cases}$

(1) $\times c$, $acx+bcy=cm.$ (3)

(2) $\times a$, $acx+ady=an.$ (4)

(3) $-$ (4), $(bc-ad)y=cm-an.$

Whence, $y=\frac{cm-an}{bc-ad}.$

(1) $\times d$, $adx+bdy=dm.$ (5)

(2) $\times b$, $bcx+bdy=bn.$ (6)

(5) $-$ (6), $(ad-bc)x=dm-bn.$

Whence, $x=\frac{dm-bn}{ad-bc}.$

**8.** $\begin{cases} mx-ny=r, \\ px-qy=s. \end{cases}$

**9.** $\begin{cases} ax+by=c, \\ mx-ny=o. \end{cases}$

**10.** $\begin{cases} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}-\frac{1}{y}=b. \end{cases}$

**11.** $\begin{cases} \frac{a}{x}+\frac{b}{y}=c, \\ \frac{a}{x}-\frac{b}{y}=d. \end{cases}$

**12.** $\frac{x}{a+b}+\frac{y}{a-b}=2a, \quad \frac{x-y}{4ab}=1.$

### EQUATIONS OF THREE OR MORE UNKNOWN NUMBERS

**212.** If there are three equations, (1), (2), (3), involving three unknown numbers,

(*a*) Combine two of the equations, as (1) and (2), so as to eliminate one of the unknown numbers, and call the resulting equation (6).

(*b*) Combine the remaining equation, (3), with either of the other two, as (1) or (2), so as to eliminate the same unknown number, and call the resulting equation (9).

(*c*) Combine (6) and (9) so as to find the unknown numbers which they involve.

(*d*) Substitute the values thus found in (1), (2), or (3), and thus find the remaining unknown number.

Find the values of $x$, $y$, and $z$ in:

$$\begin{cases} 2x + 3y + 4z = 16 & (1) \\ 3x + 2y - 5z = 8 & (2) \\ 5x - 6y + 3z = 6 & (3) \end{cases}$$

It is immaterial which unknown number we eliminate first, but there will be less work in eliminating the one which requires the least number of multiplications or the smallest multipliers to make its coefficients equal. For instance, it is best to eliminate $y$ first in the preceding example.

| | | |
|---|---|---|
| (1) $\times$ 2 | $4x + 6y + 8z = 32$ | (4) |
| (3) is | $5x - 6y + 3z = 6$ | (5) |
| (4) + (5) | $9x + 11z = 38$ | (6) |
| (2) $\times$ 3 | $9x + 6y - 15z = 24$ | (7) |
| (3) is | $5x - 6y + 3z = 6$ | (8) |
| (7) + (8) | $14x - 12z = 30$ | (9) |
| (6) $\times$ 14 | $126x + 154z = 532$ | (10) |
| (9) $\times$ 9 | $126x - 108z = 270$ | (11) |
| (10) $-$ (11) | $262z = 262$ | |
| | $\therefore \quad z = 1$ | |
| Substitute 1 for $z$ in (11), | $x = 3$ | |
| Substitute 1 for $z$ and 3 for $x$ in (1), | $y = 2.$ | |

**213.** If four or more equations are given, involving four or more unknown numbers, we proceed in a similar manner. Thus: eliminate one of the unknown numbers between three or more pairs of the equations; then a second between the different pairs of the resulting equations; and so on.

Solve the following:

**1.** $\begin{cases} 2x - y + z = 9, \\ x - 2y + 3z = 14, \\ 3x + 4y - 2z = 7. \end{cases}$

**2.** $\begin{cases} 4x - 3y + 2z = 40, \\ 5x + 9y - 7z = 47, \\ 9x + 8y - 3z = 97. \end{cases}$

**3.** $\begin{cases} 2x - 3y + 5z = 15, \\ 3x + 2y - z = 8, \\ -x + 5y + 2z = 21. \end{cases}$

**4.** $\begin{cases} 5x - 6y + 4z = 15, \\ 7x + 4y - 3z = 19, \\ 2x + y + 6z = 46. \end{cases}$

**5.** $\begin{cases} 2x - 4y + 9z = 28, \\ 7x + 3y - 5z = 3, \\ 9x + 10y - 11z = 4. \end{cases}$

**6.** $\begin{cases} 2x + 3y - 4z = 10, \\ -4y + 5z + 3x = 14, \\ 6z + 4x - 5y = 18. \end{cases}$

**7.** $\begin{cases} 3x - 3y + z = 0, \\ 2x - 7y + 4z = 0, \\ 9x + 5y + 3z = 28. \end{cases}$

**8.** $\begin{cases} x + y + z = 29, \\ x + 2y + 3z = 62, \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 10. \end{cases}$

**9.** $\begin{cases} 7x - 3y = 1, \\ 11z - 7u = 1, \\ 4z - 7y = 1, \\ 19x - 3u = 1. \end{cases}$

**10.** $\begin{cases} x + y + z + u = 14, \\ 2x + 3y + 4z + 5u = 54, \\ 4x - 5y - 7z + 9u = 10, \\ 3x + 4y + 2z - 3u = 11. \end{cases}$

**11.** $\begin{cases} ay + bx = c, \\ cx + az = b, \\ bz + cy = a. \end{cases}$

**12.** $\begin{cases} ax + by + cz = n, \\ a^2x + b^2y + c^2z = n^2, \\ a^3x + b^3y + c^3z = n^3. \end{cases}$

**13.** $\begin{cases} bx + ay = ab, \\ cx + az = ac, \\ cy + bz = bc, \end{cases}$

**14.** $\begin{cases} x + y + z = a + b + c, \\ bx + cy + az = a^2 + b^2 + c^2, \\ cx + ay + bz = a^2 + b^2 + c^2. \end{cases}$

### PROBLEMS

1. Find two numbers such that 4 times the first plus $\frac{1}{3}$ of the second is 53, and 3 times the second plus $\frac{1}{4}$ of the first is 48.

2. Find three numbers such that the first plus the second plus the third is 9; the first plus 2 times the second plus 3 times the third is 22; and the first plus 4 times the second plus 9 times the third is 58.

3. Find three numbers such that 2 times the first plus 3 times the second equal 23, 3 times the first plus 4 times the third equal 36, and 4 times the second plus 5 times the third equal 50.

4. If the cost of 7 sweaters and 4 bats is \$11.50, 5 sweaters and 6 balls is \$8.40, and 3 bats and 10 balls is \$2.25, find the prices of the sweaters, bats, and balls.

5. A said to B, "Give me 12¢ and I shall have 6 times as much as you have left"; B said to C, "Give me 24¢, and I shall have 10 times as much as you have left"; C said to A, "Give me 2¢, and I shall have 3 times as much as you have left." How much money had each?

6. A grocer sold to one customer 5 bu. wheat, 2 bu. corn, and 3 bu. rye for \$6.60; to another, 2 bu. wheat, 3 bu. corn, 5 bu. rye, for \$5.80; and to another, 3 bu. wheat, 5 bu. corn, 2 bu. rye, for \$5.60. Find the price per bushel of wheat, corn, and rye.

7. Find three numbers such that the quotient of the first by the second is 2 and the remainder 7; the quotient of the second by the third is 1 and the remainder 3; and the quotient of the first by the third is 4 and the remainder 1.

**8.** A and B can do a certain work in 24 da., B and C in 40 da., A and C in 30 da. In how many days can each do the work?

Let $x$ = the number of days it will take A to do the work.
$y$ = the number of days it will take B to do the work.
$z$ = the number of days it will take C to do the work.

Then, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, respectively, will denote the part each can do in one day.

Hence, $$\frac{1}{x}+\frac{1}{y}=\frac{1}{24}=\frac{5}{120}. \quad (1)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{40}=\frac{3}{120}. \quad (2)$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{30}=\frac{4}{120}. \quad (3)$$

(1) + (2) + (3), $$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{12}{120}. \quad (4)$$

(1) × 2, $$\frac{2}{x}+\frac{2}{y}\quad\quad=\frac{10}{120}. \quad (5)$$

(4) − (5), $\frac{2}{z}=\frac{2}{120}$; whence, $z = 120$.

(4) − 2 × (2), $\frac{2}{x}=\frac{6}{120}$; whence, $x = 40$.

(4) − 2 × (3) $\frac{2}{y}=\frac{4}{120}$; whence, $y = 60$.

**9.** A and B can do a certain work in $4\frac{4}{5}$ da., A and C in 6 da., and B and C in 8 da. In how many days can each do the work?

**10.** A, with the help of B for 4 da., can do a certain work in 15 da.; B, with the help of C for 6 da., in 20 da.; C, with the help of A for 7 da., in 22 da. How long will it take each alone to do the work?

**11.** A cistern has three pipes A, B, and C. A and B will fill it in $a$ minutes; A and C in $b$ minutes; B and C in $c$ minutes. How long will it take each alone to fill it?

**12.** A certain number is expressed by three digits. The sum of the digits is 21. The sum of the first and last digits is twice the middle digit. If the hundreds' and tens' digits are interchanged, the number is diminished by 90. Find the number.

Let $x$ = the hundreds' digit,
$y$ = the tens' digit,
$z$ = the units' digit.

Then, $100x + 10y + z$ = the number.

By the conditions, $x + y + z = 21$, (1)

$x + z = 2y$, (2)

and $100y + 10x + z = 100x + 10y + z - 90$. (3)

Solving these equations, $x = 8$, $y = 7$, $z = 6$.

Therefore, the number is 876.

**13.** A number is expressed by three digits. The middle digit is twice the left-hand digit, and one less than the right-hand digit. If 297 is added to the number, the order of the digits will be reversed. Find the number.

**14.** A number is expressed by three digits. The sum of the digits is 18; the number is equal to 99 times the sum of the first and third digits, and if 693 is subtracted from the number, the order of the digits will be reversed. Find the number.

**15.** The fore wheel of a carriage makes six revolutions more than the hind wheel in going 120 yd.; if the circumference of the fore wheel is increased by $\frac{1}{4}$ of its present size, and the circumference of the hind wheel by $\frac{1}{5}$ of its present size, the six will be changed to four. Required the circumference of each wheel.

**16.** A sum of money at simple interest amounted to \$8910 in 2 yr., and to \$9240 in 3 yr. Find the principal and the rate of interest.

**17.** A can row a boat downstream at the rate of 9 mi. per hour, and upstream at the rate of 3 mi. per hour. At what rate does the stream flow, and at what rate could A row the boat in still water?

Let $x$ = A's rate per hour in still water,
and $y$ = the rate per hour of the stream.
Then, $x + y$ = A's rate per hour downstream,
and $x - y$ = A's rate per hour upstream.
$\therefore$ $x + y = 9$,
$x - y = 3$,
Whence, $x = 6$, $y = 3$.

**18.** A crew that can pull at the rate of 15 mi. an hour downstream, finds that it takes 3 times as long to come up the river as to go down. At what rate does the stream flow?

**19.** A man sculls down a stream, which runs at the rate of 3 mi. an hour, for a certain distance in 3 hr., and finds that it takes him 12 hr. to return. Find the distance he pulled down the stream and the rate of his pulling.

**20.** A boatman can row downstream a distance of $a$ mi. and back again in $b$ hr.; and he finds that he can row $d$ mi. against the current in the same time he rows $c$ mi. with it. Find the time required in going and returning; also the rate of the current.

**21.** Two trains, whose respective lengths are 1320 ft. and 1056 ft., run on parallel tracks; when moving in opposite directions the trains pass each other in 30 sec.; when moving in the same direction, each at the same rate as before, the faster passes the slower in $1\frac{1}{2}$ min. Find the rate per hour of each train.

## INDETERMINATE EQUATIONS

**214.** Show, by substitution, that $2x + 3y = 13$ has the following pairs of values: $x = -2$, $y = 5\frac{2}{3}$; $x = -1$, $y = 5$; $x = 0$, $y = 4\frac{1}{3}$; $x = 1$, $y = 3\frac{2}{3}$; $x = 2$, $y = 3$; $x = 3$, $y = 2\frac{1}{3}$; $x = 4$, $y = 1\frac{2}{3}$; $x = 5$, $y = 1$; ...

How many of these pairs of values are positive integers?

**215.** An **indeterminate equation**, such as the one just considered, is an equation that has an unlimited number of solutions.

Generally, the object in solving indeterminate equations and problems is to find only the positive integral values of the unknown members.

Solve in positive integers, $4x + 9y = 115$. (1)

Solve for $x$, $$x = \frac{115 - 9y}{4} = 28 - 2y + \frac{3 - y}{4}. \quad (2)$$

Since the values of $x$ and $y$ are to be integral, then $\frac{3 - y}{4}$ will be an integer, otherwise we should have an integer equal to a fraction.

Hence, we may write $\frac{3 - y}{4} = n$, an integer.

Then, $3 - y = 4n$, or $y = 3 - 4n$. (3)

Substitute in (2), $x = 22 + 9n$. (4)

In (3) and (4), $n$ may have any integral value, except such as produce negative values of $x$ or $y$.

Hence, in (3), $n$ can have no positive value greater than 0, but may have any negative value.

But we see from (4) that if the absolute value of $n$ exceeds 2 ($n$ being negative), $x$ will be negative.

Therefore, $n$ can have but three values: 0, $-1$, $-2$.

Substituting these values in (3) and (4), we have

$$y = 3,\ 7,\ 11;$$
$$x = 22,\ 13,\ 4.$$

**216.** In solving equations like the preceding it is desirable that the coefficient of $y$ should be *one more* than some multiple of the coefficient of $x$, as in (1). If this relation does not exist, the equation should be multiplied by such a number as will make the new coefficient of $y$ *one more* than some multiple of the original coefficient of $x$. Then proceed as in the solution of the next example.

Solve in positive integers $5x - 14y = 11$. (1)

Here, $14 \div 5 = 2$ quo. and 4 rem., which is not in the desired form.

Multiply (1) by 4, and we have $4 \times 5x - 56y = 44$. (2)

Now, $56 \div 5 = 11$ quo. and 1 rem., which is in the desired form. It is useless to consider 4, as it is a factor of all the terms.

Transpose, divide by 5,

$$4x = \frac{44 + 56y}{5} = 8 + 11y + \frac{4 + y}{5}. \quad (3)$$

Make $\frac{4 + y}{5} = n$, whence, $y = 5n - 4$. (4)

Substitute in (3), $x = 14n - 9$. (5)

Here it is evident that $n$ may have any positive value.

| | |
|---|---|
| Making | $n = 1, 2, 3$, etc., |
| we have | $y = 1, 6, 11$, etc.; |
| and | $x = 5, 19, 33$, etc. |

Solve in positive integers:

1. $3x + 7y = 29$.
2. $4x - 7y = 13$.
3. $5x + 11y = 61$.
4. $6x - 5y = 20$.
5. $7x + 12y = 88$.
6. $13x - 9y = 103$.
7. $x + 13y = 28$.
8. $5x - 13y = 1$.
9. $23x + 17y = 47$.
10. $43x - 35y = -19$.

**11.** What number is it, which when divided by 13 the remainder is 8, and when divided by 17 the remainder is 5?

Let $m$ denote the number, then

$$\frac{m-8}{13} = x, \quad \text{and} \quad \frac{m-5}{17} = y,$$

where $x$ and $y$ are whole numbers.

Whence, $m = 13x + 8$ (1) and $m = 17y + 5$. (2)

$\therefore$ $13x + 8 = 17y + 5$. (3)

$13x = 17y - 3$. (4)

(4) $\times$ 10, $10 \times 13x = 170y - 30$. (5)

(5) $\div$ 13, $10x = 13y - 2 + \frac{y-4}{13}$. (6)

Make $\frac{y-4}{13} = n$, whence, $y = 13n + 4$.

Substitute in (2), $m = 221n + 73$.

In this $n$ may have any positive value; hence, the number is 73, or 294, or 515, etc.

**12.** The sum of three positive integers is 11; and if the first is multiplied by 3, the second by 5, and the third by 7, the sum of the products will be 57. What are the numbers?

Let $x$ = the first, $y$ = the second, $z$ = the third.

Then $x + y + z = 11$, (1)

and $3x + 5y + 7z = 57$. (2)

(1) $\times$ 7 $-$ (2), $4x + 2y = 20$,

$2x + y = 10$,

$y = 10 - 2x$.

Here $x$ may have any value from 1 to 4 inclusive.

Making $x = 1, 2, 3, 4$;

then $y = 8, 6, 4, 2$;

and $z = 2, 3, 4, 5$.

1. A farmer received \$29 for a number of lambs and pigs, receiving \$5 each for the lambs and \$2 each for the pigs. How many of each did he sell?

2. A lady spent \$2.65 in purchasing sugar at 6 cents a pound and coffee at 19 cents a pound. How many pounds of each did she buy?

3. In how many ways may a debt of \$19 be paid with 2-dollar and 3-dollar bills?

4. Seventy is divided into three parts which are divisible by 6, 7, 8, respectively, and when so divided the sum of the quotients is 10. Find the three parts.

5. Find the least number which when divided by 7 the remainder is 2, and when divided by 5 the remainder is 3.

6. The numerator of a fraction is a multiple of 2, the denominator a multiple of 3, and if 5 is added to both terms, the value of the fraction will be $\frac{8}{7}$. Find the fraction.

7. What is the least possible number of oranges on a tree, if in counting them 11 at a time there are 3 over; and in counting them 7 at a time there are 2 over?

8. A man has a number of \$10, \$5, \$1 pieces. How many of each must he take to pay a debt of \$120, using forty pieces in all?

9. A farmer buys sheep, pigs, and hens. The whole number bought is 100, and the whole cost \$100. If the sheep cost \$10, the pigs \$$2\frac{1}{2}$, and the hens \$$\frac{1}{2}$ each, how many of each did he buy?

## XI. LAWS OF OPERATIONS

(REVIEW AND EXTENSIONS)

**217. Laws of addition.**

I. *The addends may be taken in any order.*

Thus, $5+7+9=5+9+7=9+7+5.$

Or, $a+b+c=a+c+b=c+b+a.$

This is called the **commutative law** of addition.

II. *The addends may be grouped in any order.*

Thus, $5+7+9=(5+9)+7=5+(7+9).$

Or, $a+b+c=(a+b)+c=a+(b+c)=(a+c)+b.$

This is called the **associative law** of addition.

**218. Laws of multiplication.**

I. *The factors may be taken in any order.*

Thus, $3\times5\times6=3\times6\times5=6\times5\times3.$

Or, $a\times b\times c=a\times c\times b=b\times a\times c.$

This is called the **commutative law** of multiplication.

II. *The factors may be grouped in any order.*

Thus, $3\times5\times6=(3\times5)\times6=3\times(5\times6)=5\times(6\times3).$

Or, $a\times b\times c=(a\times b)\times c=a\times(b\times c)=(a\times c)\times b.$

This is called the **associative law** of multiplication.

III. The **distributive law** of multiplication, viz.:

$$(a+b+c)\times n=an+bn+cn. \quad \S\ 33.$$

The above laws of addition and multiplication are true for any number of addends or factors.

### LAWS OF POSITIVE INTEGRAL EXPONENTS

**219.** In all that follows $a$, $b$, $c$ stand for any real numbers, and $m$, $n$, $p$ for any positive integers.

**Definition.** $a^n$ or $(a)^n = a \times a \times \cdots$ to $n$ factors, where $a$ is the *base*, $n$ the *exponent*, and $a^n$ the $n$th *power* of $a$.

When $a$ is negative, $a^n$ is positive or negative according as $n$ is even or odd.

**220.** I. *The product of the mth and nth powers of any number is equal to the $(m+n)$th power of that number.*

PROOF.

$$a^m \times a^n = (a \times a \times \cdots \text{to } m \text{ factors}) \times (a \times a \times \cdots \text{to } n \text{ factors})$$
$$= a \times a \times a \cdots \text{to } (m+n) \text{ factors}$$
$$= a^{m+n}.$$

EXAMPLES. $x^5 \times x^3 = x^8$, $(a+b)^6 \times (a+b)^7 = (a+b)^{13}$.

Evidently, $a^m \times a^n \times a^p \times \cdots = a^{m+n+p+\cdots}$.

**221.** II. *The quotient of the mth by the nth power of any number, m being greater than n, is equal to the $(m-n)$th power of that number.*

PROOF. By § 220, $a^{m-n} \times a^n = a^{m-n+n} = a^m$.

$\therefore \quad a^m \div a^n = a^{m-n}$. § 153, (2)

EXAMPLES. $x^9 \div x^4 = x^5$, $(a-b)^7 \div (a-b)^3 = (a-b)^4$.

**222.** III. *The mth power of the nth power of any number is equal to the mnth power of that number.*

PROOF. $(a^n)^m = a^n \times a^n \times \cdots$ to $m$ factors

$$= a^{n+n+\cdots \text{ to } m \text{ terms}}$$
$$= a^{mn}.$$

EXAMPLES. $(x^3)^2 = x^6$; the 4th power of $a^5$ is $a^{20}$.

**223.** IV. *The nth power of the product of several numbers is equal to the product of the nth powers of those numbers.*

PROOF.

$(ab)^n = ab \times ab \times \cdots$ to $n$ factors

$= (a \times a \times \cdots$ to $n$ factors$)(b \times b \times \cdots$ to $n$ factors$)$

$= a^n b^n$.

EXAMPLES. $6^3 = 2^3 \times 3^3$, $(a^4b^5)^2 = a^8b^{10}$.

**224.** V. *The nth power of the quotient of any two numbers is equal to the quotient of the nth powers of those numbers.*

PROOF. $\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \cdots$ to $n$ factors

$$= \frac{a \times a \times \cdots \text{ to } n \text{ factors}}{b \times b \times \cdots \text{ to } n \text{ factors}} = \frac{a^n}{b^n}.$$

EXAMPLES. $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$; $\left(\frac{a^2}{b^3}\right)^3 = \frac{(a^2)^3}{(b^3)^3} = \frac{a^6}{b^9}$.

## LAWS OF POSITIVE INTEGRAL INDEXES

**225.** In all that follows $a$, $b$, $c$ stand for any real numbers, and $m$, $n$, $p$ for any positive integers.

**Definition.** The $n$th root of $a$, denoted by $\sqrt[n]{a}$, is one of the $n$ equal factors of $a$, $n$ being the *index* of the root.

That is, $\sqrt[n]{a} \times \sqrt[n]{a} \times \cdots$ to $n$ factors $= a$.

When the index ($n$) is even, and $a$ is negative, $\sqrt[n]{a}$ is *imaginary* (§ 267).

**226.** I. *The nth root of the mnth power of any real number is the mth power of that number.*

PROOF. By § 222, $(a^m)^n = a^{mn}$.

$\therefore \quad \sqrt[n]{a^{mn}} = a^m$. § 153, (3)

That is, $\sqrt[n]{a^{mn}} = a^{mn \div n} = a^m$.

This law does not hold when $a$ is negative if $n$ is even and $m$ is odd.

For example, $\sqrt[2]{(-3)^6}$ is not equal to $(-3)^3$ or $-27$, since $(-3)^6 = (+3)^6$, and hence $\sqrt{(-3)^6} = \sqrt{(3)^6} = 3^3 = +27$.

**227.** II. *The product of the nth roots of any two real numbers is equal to the nth root of the product of those numbers.*

PROOF. $(\sqrt[n]{a} \times \sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n$ § 223

$= ab.$

$\therefore \quad \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}.$ § 153, (3)

This law does not hold when $a$ and $b$ are both negative.

For example, it is not true that $\sqrt{-8} \times \sqrt{-2} = \sqrt{-8 \times -2} = \sqrt{16} = 4$. For, $\sqrt{-8} = \sqrt{8}\sqrt{-1}$ and $\sqrt{-2} = \sqrt{2}\sqrt{-1}$, and $\sqrt{8}\sqrt{-1} \times \sqrt{2}\sqrt{-1} = \sqrt{16}(\sqrt{-1})^2 = 4 \times -1 = -4$.

**228.** III. *The quotient of the nth roots of any two real numbers is equal to the nth root of the quotient of those numbers.*

PROOF. $\left(\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n = \dfrac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n} = \dfrac{a}{b}.$ § 224

$\therefore \quad \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}.$ § 153, (3)

Reduce the following, and give the laws involved:

1. $x^5 \times x^3$.
2. $2^8 \times 2^2$.
3. $3^4a^6 \times 3^2a^2$.
4. $x^5 \div x^3$.
5. $2^8 \div 2^2$.
6. $3^4a^6 \div 3^2a^2$.
7. $(x^3)^4$.
8. $(3^2)^3$.
9. $(2^2a^3b^2c)^2$.
10. $(-a^2)^3$.
11. $((-2)^2)^4$.
12. $(-5x^2y^4)^3$.
13. $\left(\dfrac{a^2}{b^3}\right)^5$.
14. $\left(-\dfrac{2a^2}{3b^3}\right)^6$.
15. $\left(\dfrac{a^2c^3x^4}{b^5y^6}\right)^4$.
16. $\sqrt{x^{10}}$.
17. $\sqrt[3]{-8a^6b^3}$.
18. $\sqrt[5]{-a^5b^{10}x^{15}}$.
19. Find the 2d power of $2^3x^2y \times 3^2xy^4$.
20. Find the 3d power of $3^5a^3c^4 \div 3^3ac$.
21. Find the 4th root of $5^3a^2b \times 5^5a^2b^7$.
22. Find the 5th root of $3^9a^{12}b^8 \div 3^4a^2b^3$.

## POWERS OF BINOMIALS

**229.** By actual multiplication it is found that:

$$(a+b)^1 = a + b.$$
$$(a+b)^2 = a^2 + 2ab + b^2.$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

From these results it will be noted that:

I. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

II. In the first term, the exponent of $a$ is the same as the exponent of the power to which the binomial is raised; and it decreases by one in each succeeding term.

III. $b$ appears in the second term with 1 for an exponent, and its exponent increases by one in each succeeding term.

IV. The coefficient of the first term is 1.

V. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

VI. The coefficient of each succeeding term is found from the next preceding term by multiplying its coefficient by the exponent of $a$, and dividing the product by a number greater by one than the exponent of $b$.

If $b$ is negative, the terms in which the odd powers of $b$ occur are negative. Thus:

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

This process of writing out the power of a binomial is called **expanding** the binomial.

Expand the following:

| | | |
|---|---|---|
| 1. $(a+x)^3=$ | 2. $(a+x)^4=$ | 3. $(x+y)^5=$ |
| 4. $(a-x)^3=$ | 5. $(x-a)^4=$ | 6. $(x-y)^5=$ |
| 7. $(a+1)^3=$ | 8. $(x+1)^4=$ | 9. $(x+1)^5=$ |
| 10. $(x-1)^3=$ | 11. $(x-1)^4=$ | 12. $(y-1)^5=$ |

**230.** The same method may be employed when the terms of a binomial have coefficients or exponents.

Let it be observed that the following expansions are in accordance with the preceding rule, provided the first term of the binomial is regarded as (or substituted for) $a$, and the second term of the binomial for $b$.

$$(3x-y^2)^3 = (3x)^3 - 3(3x)^2(y^2) + 3(3x)(y^2)^2 - (y^2)^3$$
$$= 27x^3 - 27x^2y^2 + 9xy^4 - y^6.$$
$$(c^2 - \tfrac{1}{2}x)^4 = (c^2)^4 - 4(c^2)^3(\tfrac{1}{2}x) + 6(c^2)^2(\tfrac{1}{2}x)^2 - 4(c^2)(\tfrac{1}{2}x)^3 + (\tfrac{1}{2}x)^4 =$$
$$c^8 - 2c^6x + \tfrac{3}{2}c^4x^2 - \tfrac{1}{2}c^2x^3 + \tfrac{1}{16}x^4.$$

Similarly, a polynomial of three or more terms may be raised to any power by first writing it under the form of a binomial. Thus:

$$(x^2+2x+3)^3 = \{x^2 + (2x+3)\}^3$$
$$= (x^2)^3 + 3(x^2)^2(2x+3) + 3(x^2)(2x+3)^2 + (2x+3)^3$$
$$= x^6 + 6x^5 + 21x^4 + 44x^3 + 63x^2 + 54x + 27.$$

Simplify or expand the following:

| | | |
|---|---|---|
| 13. $(x+3)^3$. | 14. $(x^2-2a)^3$. | 15. $(a^2+a+1)^3$. |
| 16. $(x-2)^3$. | 17. $(a^2-3a)^4$. | 18. $(x^2-x-1)^2$. |
| 19. $(a+4)^4$. | 20. $(a^3+4c)^3$. | 21. $(1-2a+a^2)^4$. |
| 22. $(1+2a)^5$. | 23. $(2a^2-3y)^4$. | 24. $(5-2x-x^2)^3$. |
| 25. $(2a-1)^5$. | 26. $(2a^3-\frac{1}{2}x)^5$. | 27. $(x^2-5x+2)^3$. |

## CUBE ROOT

**231.** The **cube root** of an integer is one of its three equal factors.

Name the cube roots of the following:

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

**232.** A **perfect cube** is a number that is the product of three equal integers or fractions; and an **imperfect cube** is a number that is not the product of three equal integers or fractions.

The cube root of a number is indicated by attaching the vinculum to the radical sign and placing the figure 3 in the opening, $\sqrt[3]{\ }$ (§ 139).

Thus, $\sqrt[3]{27\,a^6b^3} = 3\,a^2b$, and $\sqrt[3]{(a^3 + 3\,a^2b + 3\,ab^2 + b^3)} = a + b$.

**233. Finding the cube roots of monomials.**

Find the cube root of $125\,a^6b^9$.

$\sqrt[3]{125} = 5$, $\sqrt[3]{a^6} = a^2$, $\sqrt[3]{b^9} = b^3$. $\therefore\ \sqrt[3]{125\,a^6b^9} = 5\,a^2b^3$.

*Extract the cube root of the numerical coefficient, and divide the exponent of each literal factor by* 3.

Find the values of:

1. $\sqrt[3]{27\,a^{12}}$.
2. $\sqrt[3]{m^9n^6x^3}$.
3. $\sqrt[3]{1000\,x^3y^{12}}$.
4. $\sqrt[3]{64\,n^{15}}$.
5. $\sqrt[3]{216\,a^{24}}$.
6. $\sqrt[3]{729\,a^6c^{36}}$.
7. $\sqrt[3]{8\,x^3y^6}$.
8. $\sqrt[3]{512\,x^3y^9}$.
9. $\sqrt[3]{343\,a^9n^{27}}$.
10. $\sqrt[3]{a^3(a+x)^6}$.
11. $\sqrt[3]{8(2+x)^3(3-x)^6}$.
12. $\sqrt[3]{27(x-y)^3}$.
13. $\sqrt[3]{n^3(x+a)^9(x-6)^{12}}$.
14. $\sqrt[3]{64(a-b)^6}$.
15. $\sqrt[3]{a^6(c+d)^3(x-y)^9}$.

**234. Extracting the cube roots of polynomials.**

Since the cube of $t + o = t^3 + 3\,t^2o + 3\,to^2 + o^3$ (§ 229), the cube root of the latter is the former.

Now we wish to determine how $t + o$ may be found when $t^3 + 3\,t^2o + 3\,to^2 + o^3$ is known, whatever the values of $t$ and $o$ may be.

$$\begin{array}{l|l} t^3 + 3\,t^2o + 3\,to^2 + o^3 & t + o \\ \underline{t^3} & \\ & 3\,t^2o + 3\,to^2 + o^3 \\ & \underline{3\,t^2o + 3\,to^2 + o^3} \end{array}$$

Trial divisor, $3\,t^2$

Complete divisor, $3\,t^2 + 3\,to + o^2$

Evidently, the first term, $t$, of the root is the cube root of the first term, $t^3$, of the given number.

Subtracting $t^3$ from the given number leaves the remainder $3\,t^2o + 3\,to^2 + o^3 = (3\,t^2 + 3\,to + o^2)\,o$. Now $3\,t^2$, *3 times the square of the part of the root already found*, or 3 times the square of the first part of the root, is the **trial divisor.** Dividing the first term of the remainder, $3\,t^2o$, by the trial divisor gives the quotient $o$, the second part of the root.

The **complete divisor** is formed by annexing or adding to the trial divisor $3\,to + o^2$; that is, *3 times the product of the first part of the root by the second part plus the square of the second part.*

Find the cube root of $8\,a^6 - 60\,a^4c + 150\,a^2c^2 - 125\,c^3$.

$$\begin{array}{l|l} 8\,a^6 - 60\,a^4c + 150\,a^2c^2 - 125\,c^3 & 2\,a^2 - 5\,c \\ \underline{8\,a^6} & \\ & -60\,a^4c + 150\,a^2c^2 - 125\,c^3 \\ & \underline{-60\,a^4c + 150\,a^2c^2 - 125\,c^3} \end{array}$$

T. divisor $= 12\,a^4$

C. divisor $= 12\,a^4 - 30\,a^2c + 25\,c^2$

In this example $2\,a^2$ and $-5\,c$ correspond to $t$ and $o$ in the typical example given above, $2\,a^2$ being the first part of the root and $-5\,c$ the second part.

The cube root of the first term, $8\,a^6$, is $2\,a^2$, which is the first term of the root. The cube of $2\,a^2$ subtracted from the given number leaves the remainder $-60\,a^4c + 150\,a^2c^2 - 125\,c^3$.

Three times the square of $2\,a^2$ is $12\,a^4$, which is the trial divisor. Dividing $-60\,a^4c$ by $12\,a^4$ gives the quotient $-5\,c$, which is the second term of the root. To form the complete divisor we annex to the trial divisor $3 \times 2\,a^2 \times -5\,c$ plus $(-5\,c)^2 = -30\,a^2c + 25\,c^2$, which gives the complete divisor $12\,a^4 - 30\,a^2c + 25\,c^2$, which, when multiplied by $-5\,c$, exhausts the remainder.

Find the cube roots of:

1. $x^3 + 12x^2 + 48x + 64$.
2. $8a^3 + 36a^2 + 54a + 27$.
3. $8x^3 - 84x^2 + 294x - 343$.
4. $1 + 9m + 27m^2 + 27m^3$.
5. $a^6 - 15a^5 + 75a^4 - 125a^3$.
6. $8a^3c^3 - 12a^2bc^2 + 6ab^2c - b^3$.
7. $64a^3 - 144a^2c + 108ac^2 - 27c^3$.
8. $343x^3 + 588x^2y + 336xy^2 + 64y^3$.
9. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1 \mid x^2 - 2x + 1$

$x^6$

| | |
|---|---|
| T.D. $= 3x^4$ | $-6x^5 + 15x^4 - 20x^3$ |
| C.D. $= 3x^4 - 6x^3 + 4x^2$ | $-6x^5 + 12x^4 - 8x^3$ |
| T.D. $= 3x^4 - 12x^3 + 12x^2$ | $3x^4 - 12x^3 + 15x^2 - 6x + 1$ |
| C.D. $= 3x^4 - 12x^3 + 15x^2 - 6x + 1$ | $3x^4 - 12x^3 + 15x^2 - 6x + 1$ |

Having found the first two terms, $x^2 - 2x$, of the root as in the preceding examples, we consider $x^2 - 2x$ the first term or part of the root. The next trial divisor will then be $3(x^2 - 2x)^2$, or $3x^4 - 12x^3 + 12x^2$, and the second term of the root, $3x^4 \div 3x^4$, 1. To the trial divisor we add $3(x^2 - 2x) \times 1$ plus $(1)^2$ or $3x^2 - 6x + 1$, and obtain the above complete divisor.

If we call the part of the root already found the first part and the next term to be found the second part, at any stage of the process, the trial divisor is 3 times the square of the first part, and the complete divisor is the trial divisor, plus 3 times the product of the first part by the second, plus the square of the second part.

In like manner find the cube roots of:

10. $a^6 - 9a^5 + 33a^4 - 63a^3 + 66a^2 - 36a + 8$.
11. $x^6 - 3x^5 + 5x^3 - 3x - 1$.
12. $x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1$.

**235. Finding the cube roots of arithmetical numbers.**

To extract the cube root of an arithmetical number we first determine how many figures or digits there will be in the root, and what the first digit will be, which we can do by inspection as follows. Note that each number is divided into periods of three figures each, beginning at the right.

Since 42'875 lies between 27'000 and 64'000, its cube root lies between 30 and 40. Hence, there are two figures in the cube root of 42'875, and the first figure, 2, of the root is the cube root of the greatest cube contained in 42.

Since 187'149'248 lies between 125'000'000 and 216'000'000, its cube root lies between 500 and 600. Hence, there are three figures in the cube root of 187'149'248, and the first figure, 5, of the root is the cube root of the greatest cube contained in 187.

Compare the number of periods in the following cubes and the number of figures in the cube roots:

| Number | Root | Number | Root | Number | Root |
|---|---|---|---|---|---|
| 1 | 1 | 1'000 | 10 | 1'000'000 | 100 |
| 729 | 9 | 970'299 | 99 | 997'002'999 | 999 |

From these, and similar examples, we infer:

*There are as many figures in the cube root of a perfect cube as there are 3-figure periods in the number.*

*The first figure of the root is the cube root of the greatest cube contained in the left-hand period.*

Note. The left-hand period may consist of only one or two figures.

Give the first figure and the number of figures in the cube roots of:

| | | |
|---|---|---|
| 1. 46'656. | 2. 91'125. | 3. 127'263'527. |
| 4. 274'625. | 5. 124'000. | 6. 743'677'416. |
| 7. 681'472. | 8. 238'328. | 9. 16'581'375. |
| 10. 343'000. | 11. 8'741'816. | 12. 1'879'080'904. |

Find the cube root of 46,656.

Since there are two periods in 46′656, there will be two figures in its cube root. The tens' figure will be 3.

Denoting the tens, 30, of the root by $t$ and the ones by $o$, we have $(t+o)^3 = t^3 + 3\,t^2o + 3\,to^2 + o^3 = 46{,}656$.

Hence, the following operation is like the typical one given on page 194.

| | | | $t+o$ |
|---|---|---|---|
| | | | 46′656 \| 30 + 6 = 36 |
| | | $t^3$ = | 27 000 |
| T. divisor, | $3\,t^2$ | = 2700 | 19 656 |
| | $3\,to$ | = 540 | |
| | $o^2$ | = 36 | |
| C. divisor, | $t^2 + 3\,to + o^2$ | = 3276 | 19 656 |

The greatest cube in 46 is 27, and its cube root is 3, which is the number of tens in the root.

The trial divisor is then $3 \times (30)^2$, or 2700.

In dividing the remainder 19,656 by 2700 the quotient would appear to be 7. But when the divisor is completed, 7 proves to be too large. So we take 6 for the units' figure.

In like manner find the cube roots of:

| | | |
|---|---|---|
| 1. 13,824. | 2. 421,875. | 3. 262,144. |
| 4. 42,875. | 5. 175,616. | 6. 250,047. |
| 7. 74,088. | 8. 405,224. | 9. 912,673. |

The following solutions of Examples 8 and 9 will show how the unnecessary ciphers may be omitted and the solution abbreviated.

| | | | | | |
|---|---|---|---|---|---|
| | 405′224 | 74 | | 912′673 | 97 |
| | 343 | | | 729 | |
| T. D. | 14700 | 62 224 | | 24300 | 183 673 |
| | 840 | | | 1890 | |
| | 16 | | | 49 | |
| C. D. | 15556 | 62 224 | | 26239 | 183 673 |

Find the cube roots of:

| | | |
|---|---|---|
| 1. 54,872. | 2. 148,877. | 3. 405,224. |
| 4. 46,656. | 5. 571,787. | 6. 175,616. |
| 7. 91,125. | 8. 42,875. | 9. 110,592. |

10. 109,215,352.

$$
\begin{array}{lllll}
 & & & 109'215'352 & 478 \\
 & & & 64 & \\
 & & 4800 & 45\,215 & \\
 & & 840 & & \\
 & & 49 & & \\
 & & 5689 & 39\,823 & \\
\text{Trial divisor,} & 3(470)^2 & 662700 & 5\,392\,352 & \\
 & 3 \times 470 \times 8 & 11280 & & \\
 & 8^2 & 64 & & \\
 & \text{Complete divisor,} & 674044 & 5\,392\,352 &
\end{array}
$$

The first two figures, 47, of the root are found as in the preceding examples.

Then 47 is considered the tens of the root. Hence, the next trial divisor is $3 \times (470)^2$, and the units of the root is 8.

The complete divisor is then formed of the tens and ones as heretofore.

| | |
|---|---|
| 11. 44,361,864. | 12. 743,677,416. |
| 13. 16,581,375. | 14. 130,323,843. |

The notes on page 122 are applicable to cube root with the change of "square" to "cube."

Find the values of:

| | | |
|---|---|---|
| 15. $\sqrt[3]{.015625}$. | 16. $\sqrt[3]{.001728}$. | 17. $\sqrt[3]{.000166375}$. |
| 18. $\sqrt[3]{\frac{64}{343}}$. | 19. $\sqrt[3]{\frac{729}{1331}}$. | 20. $\sqrt[3]{372191\frac{1}{4}}$. |
| 21. $\sqrt[3]{2}$. | 22. $\sqrt[3]{3}$. | 23. $\sqrt[3]{5}$. |

### LAWS OF FRACTIONAL AND NEGATIVE EXPONENTS

**236.** The main laws for positive integral exponents are, as we have seen:

I. $a^m \times a^n = a^{m+n}$. III. $(a^m)^n = a^{mn}$.

II. $a^m \div a^n = a^{m-n}$. IV. $(a^m \times b^n)^p = a^{mp} \times b^{np}$.

**237.** It is now to be shown that these laws are true for any rational exponents, and the first step is to determine what is meant by fractional, zero, and negative exponents. For, the meaning heretofore attached to exponent, § 56, cannot apply to an exponent that is a fraction, zero, or a negative number.

Thus, with respect to $a^{\frac{2}{3}}$, $a^0$, $a^{-3}$ it is senseless to say $a$ is taken as a factor $\frac{2}{3}$ times, or 0 times, or $-3$ times.

The meaning of such exponents may be determined by assuming that Law I holds for all rational values of $m$ and $n$, provided that Laws II, III, IV (p. 188) are shown to be true in the same sense or for the same values of $m$, $n$, and $p$.

**238. Finding the meaning of a zero exponent.**

Make $m = 0$ in I, $a^m \times a^n = a^{m+n}$.

The result is $a^0 \times a^n = a^n$.

Whence, § 153, (2), $a^0 = a^n \div a^n = 1$.

*Therefore a number with a zero exponent means* **1.**

Thus, $2^0 = 1$, $x^0 = 1$, $(a + b)^0 = 1$.

The student may prove these as follows:

$$2^0 = 2^{n-n} = 2^n \div 2^n = 1;$$

$$x^0 = x^{m-m} = x^m \div x^m = 1;$$

$$(a + b)^0 = (a + b)^{p-p} = (a + b)^p \div (a + b)^p = 1.$$

**239. Finding the meaning of a negative exponent.**

Make $n = -m$ in I, $a^m \times a^n = a^{m+n}$.

The result is $a^m \times a^{-m} = a^0 = 1$.

Whence, § 153, (2), $a^{-m} = \frac{1}{a^m}$.

*Therefore, a number with a negative exponent means the reciprocal of the number with a positive exponent of the same absolute value.*

Thus, $2^{-1} = \frac{1}{2}$, $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$, $c(a+b)^{-1} = \frac{c}{a+b}$.

**240. Finding the meaning of a fractional exponent.**

Make $m = n = \frac{1}{2}$ in I, $a^m \times a^n = a^{m+n}$.

The result is $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$.

That is, $(a^{\frac{1}{2}})^2 = a$.

Whence, § 153, (3), $a^{\frac{1}{2}} = \sqrt{a}$.

In general, § 219, $(a^{\frac{m}{n}})^n = a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \cdots$ to $n$ factors

$= a^{\frac{m}{n} + \frac{m}{n} + \cdots \text{ to } n \text{ terms}}$

$= a^m$.

Whence, § 153, (3), $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

*Therefore, a number with a fractional exponent, both terms of the fraction being positive, means a root of a power of the number, the denominator being the index of the root and the numerator the exponent of the power.*

Thus, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$, $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$.

**241.** It remains now to be shown that these interpretations of zero, negative, and fractional exponents are not inconsistent with Laws II, III, IV (p. 188). To do so it will be sufficient to verify these laws according to the meanings that have been assigned to exponents in general.

**242. Verifying Law II, $a^m \div a^n = a^{m-n}$.**

By Law I, $a^{m-n} \times a^n = a^m$,

for all rational values of $m$ and $n$.

Whence, § 153, (2), $a^m \div a^n = a^{m-n}$,

for all rational exponents.

**243. Verifying Law III, $(a^n)^m = a^{mn}$.**

Let $q$ and $r$ be any positive integers, and let $k$ be any rational number. Then we have:

(1) $(a^k)^{\frac{q}{r}} = \sqrt[r]{(a^k)^q} = \sqrt[r]{a^{kq}} = a^{\frac{kq}{r}} = a^{\frac{q}{r}\times k}$, §§ 240, 222.

(2) $(a^k)^{-\frac{q}{r}} = \dfrac{1}{(a^k)^{\frac{q}{r}}} = \dfrac{1}{a^{\frac{q}{r}\times k}} = a^{-\frac{q}{r}\times k}$. §§ 239, 243, (1).

Therefore Law III is true for all rational values of $m$ and $n$.

**244. Verifying Law IV, $(a^m \times b^n)^p = a^{mp} \times b^{np}$.**

Let $q$ and $r$ be any positive integers, and let $m$ and $n$ be any rational numbers. Then we have:

(1) $(a^m b^n)^{\frac{q}{r}} = \sqrt[r]{(a^m b^n)^q} = \sqrt[r]{a^{mq} b^{nq}}$,

$= \sqrt[r]{a^{mq}} \times \sqrt[r]{b^{mq}} = a^{\frac{q}{r}\times m} \times b^{\frac{q}{r}\times n}$. §§ 240, 223.

(2) $(a^m b^n)^{-\frac{q}{r}} = \dfrac{1}{(a^m b^n)^{\frac{q}{r}}} = \dfrac{1}{a^{\frac{q}{r}\times m} b^{\frac{q}{r}\times n}} = a^{-\frac{q}{r}\times m} b^{-\frac{q}{r}\times n}$.

§§ 239, 244, (1).

Therefore Law IV is true for all rational values of $m$ and $n$.

It follows from Laws II, III, IV that any monomial is raised to any power by multiplying the exponent of each factor of the monomial by the given exponent.

Thus, $(a^{\frac{3}{4}} b^{-4} c^{-\frac{1}{2}})^{-\frac{3}{4}} = a^{\frac{3}{4}\times -\frac{3}{4}} b^{4\times -\frac{3}{4}} c^{-\frac{1}{2}\times -\frac{3}{4}} = a^{-\frac{9}{16}} b^{3} c^{\frac{3}{8}}$.

$(8 x^3 z^{-\frac{1}{2}})^{\frac{2}{3}} = 8^{\frac{2}{3}} x^2 z^{-\frac{1}{3}} = \sqrt[3]{8^2} x^2 z^{-\frac{1}{3}} = 4 x^2 z^{-\frac{1}{3}}$.

**245.** The meanings of fractional and negative exponents, as derived from Law I, hold for all the other laws. Therefore these new kinds of exponents simply provide new ways of indicating familiar operations.

Thus, for $\sqrt[5]{x^2}$ we may write $x^{\frac{2}{5}}$, and for $\frac{a}{\sqrt[4]{x^3}}$ we may write $\frac{a}{x^{\frac{3}{4}}}$, or $ax^{-\frac{3}{4}}$. That is, according to § 239, we may transfer any factor from the denominator of a fraction to the numerator, or from the numerator to the denominator, provided the sign of its exponent is changed.

### EXERCISES

Let the student name the laws that are employed in reducing the following:

1. $64^{-\frac{1}{3}} = \frac{1}{64^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}.$

2. $81^{\frac{3}{4}} = \sqrt[4]{81^3} = (\sqrt[4]{81})^3 = 3^3 = 27.$

3. $x^{\frac{2}{3}} \times x^{\frac{1}{2}} = x^{\frac{2}{3}+\frac{1}{2}} = x^{\frac{7}{6}} = \sqrt[6]{x^7}.$

4. $x^{\frac{1}{2}} \times x^{\frac{1}{3}} \times x^{-\frac{5}{6}} = x^{\frac{1}{2}+\frac{1}{3}-\frac{5}{6}} = x^0 = 1.$

5. $(b^{-\frac{3}{4}})^8 = b^{-\frac{3}{4}\times 8} = b^{-6} = \frac{1}{b^6}.$

6. $(9x^{-\frac{2}{3}})^{-\frac{3}{2}} = 9^{-\frac{3}{2}}(x^{-\frac{2}{3}\times-\frac{3}{2}}) = \frac{x^1}{9^{\frac{3}{2}}} = \frac{x}{(\sqrt{9})^3} = \frac{x}{27}.$

7. $(125\,a^{\frac{3}{4}})^{-\frac{2}{3}} = 125^{-\frac{2}{3}}(a^{\frac{3}{4}\times-\frac{2}{3}}) = \frac{1}{125^{\frac{2}{3}}a^{\frac{1}{2}}} = \frac{1}{25\sqrt{a}}.$

8. $a^{-2}b^{\frac{2}{3}}c^{-\frac{3}{4}} = \frac{b^{\frac{2}{3}}}{a^2c^{\frac{3}{4}}} = \frac{\sqrt[3]{b^2}}{a^2\sqrt[4]{c^3}}.$

Express with fractional exponents:

1. $\sqrt[3]{a}$. 2. $\sqrt[5]{x^2}$. 3. $\sqrt[6]{c}$. 4. $\sqrt[4]{a} + \sqrt[3]{c^2}$.
5. $\sqrt{a^3}$. 6. $\sqrt[8]{y^3}$. 7. $\sqrt[9]{c^5}$. 8. $\sqrt[6]{c^2} + \sqrt[5]{m}$.

Express with radical signs:

9. $a^{\frac{3}{7}}$. 10. $a^{\frac{1}{2}}b^{\frac{1}{3}}$. 11. $c^{\frac{1}{4}}x^{-\frac{1}{2}}$. 12. $a^{\frac{1}{3}} - 3b^{\frac{1}{4}}$.
13. $c^{\frac{2}{5}}$. 14. $a^{\frac{1}{4}}c^{\frac{3}{4}}$. 15. $n^{\frac{3}{5}}x^{-\frac{2}{5}}$. 16. $a^{\frac{1}{2}} + 2c^{\frac{2}{3}}$.

Express with positive exponents:

17. $a^{-2}$. 18. $3a^{-5}b^2$. 19. $bc^{-3}x^{-4}$. 20. $m^{-2}nx^{-3}$.
21. $\dfrac{b^{-3}}{c^{-4}}$. 22. $\dfrac{ab^{-2}c^3}{x^2y^{-5}}$. 23. $\dfrac{5n^{-7}x}{6a^{-3}y}$. 24. $\dfrac{a^{-2}b^{-3}c^4}{x^{-5}y^{-2}z}$.

Write without denominators:

25. $\dfrac{a^2b^{-3}}{c^4}$. 26. $\dfrac{a^2d^{-3}x^2}{b^{-5}c^4}$. 27. $\dfrac{x^3y^{-4}z^{-2}}{y^{-3}z^{-4}}$.
28. $\dfrac{a^3b^{-5}}{a^{-2}b^4}$. 29. $\dfrac{c^2d^{-4}n^{-5}}{cd^{-6}c^3}$. 30. $\dfrac{5c^2n}{a^{-4}b^{-6}c^3}$.

Find the values of:

31. $9^{\frac{5}{2}}$. 32. $(-8)^{\frac{5}{3}}$. 33. $16^{-\frac{1}{4}} \times 8^{\frac{2}{3}}$.
34. $4^{-\frac{7}{2}}$. 35. $64^{-\frac{2}{3}}$. 36. $25^{-\frac{1}{2}} \div 4^{-\frac{3}{2}}$.
37. $(-8)^{-\frac{2}{3}}$. 38. $81^{-\frac{3}{4}}$. 39. $(a^{\frac{2}{3}}c^{-\frac{1}{2}}d)^6$.
40. $a^{\frac{1}{2}} \times a^{\frac{3}{5}}$. 41. $c^{\frac{2}{3}} \times c^{-\frac{1}{2}}$. 42. $(\sqrt[4]{a^3}\sqrt[3]{c^2})^{-6}$.
43. $x^{\frac{2}{3}} \div x^{-\frac{1}{4}}$. 44. $8^{\frac{2}{3}} \div 9^{-\frac{1}{2}}$. 45. $(\sqrt[6]{a^5}\sqrt[4]{b^3})^{12}$.
46. $(\sqrt{a}\sqrt[3]{b})^6$. 47. $\sqrt[3]{c^2}\sqrt[8]{c^5}$. 48. $(x^{\frac{3}{5}}y^{-\frac{1}{2}}z)^{\frac{6}{5}}$.

If $a = 8$, $b = 4$, $c = 1$, find the value of:

49. $a^{\frac{1}{3}}b^{\frac{1}{2}}$. 50. $a^{-\frac{2}{3}}b^2$. 51. $(abc)^{\frac{2}{5}}$.
52. $ab^{-2}$. 53. $a^{\frac{5}{3}}b^{-\frac{3}{2}}$. 54. $a^{-\frac{7}{3}} \times b^{\frac{5}{2}}$.

**246. Multiplying polynomials involving fractional and negative exponents.**

Multiply $a^{\frac{3}{4}} - a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} - b^{\frac{3}{4}}$ by $a^{\frac{1}{4}} + b^{\frac{1}{4}}$.

$$\begin{array}{l} a^{\frac{3}{4}} - a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} - b^{\frac{3}{4}} \\ a^{\frac{1}{4}} + b^{\frac{1}{4}} \\ \hline a - a^{\frac{3}{4}}b^{\frac{1}{4}} + a^{\frac{1}{2}}b^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{3}{4}} \\ \quad + a^{\frac{3}{4}}b^{\frac{1}{4}} - a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{3}{4}} - b \\ \hline a \qquad\qquad\qquad\qquad - b \end{array}$$

Multiply:

1. $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.
2. $c^{\frac{1}{2}} + x^{\frac{1}{2}}$ by $c^{\frac{1}{2}} + x^{\frac{1}{2}}$.
3. $c^{\frac{1}{2}} - x^{\frac{1}{2}}$ by $c^{\frac{1}{2}} - x^{\frac{1}{2}}$.
4. $a^{\frac{1}{3}} + b^{\frac{1}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.
5. $a^{\frac{3}{4}} + a^{\frac{1}{4}}$ by $a^{\frac{3}{4}} + a^{\frac{1}{4}}$.
6. $x^{\frac{3}{4}} - x^{\frac{1}{4}}$ by $x^{\frac{1}{4}} + x^{\frac{3}{4}}$.
7. $a^{-2} - 2a^{-1}b + b^2 - ab^3$ by $a^{-3} + 2a^{-2}b$.
8. $x^{-3}y^2 - x^{-2}y - 2x^{-1}$ by $2x^2y^{-1} + 2x^3y^{-2} - 4x^4y^{-3}$.
9. $a^{\frac{1}{2}} + b^{\frac{1}{2}} + a^{-\frac{1}{2}}b$ by $ab^{-\frac{1}{2}} - a^{\frac{1}{2}} + b^{\frac{1}{2}}$.
10. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.
11. $a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b$ by $a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b$.
12. $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 4 + x^{-\frac{1}{3}}$ by $3x^{\frac{4}{3}} + x - 2x^{\frac{2}{3}}$.
13. $x^{-1} + 3x^{-\frac{2}{3}} + 3x^{-\frac{1}{3}} + 1$ by $x^{-1} - 3x^{-\frac{2}{3}} + 3x^{-\frac{1}{3}} - 1$.

Prove by multiplication that the rationalizing factor (see § 262):

14. Of $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ is $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.
15. Of $x^{\frac{1}{3}} - y^{\frac{1}{3}}$ is $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.
16. Of $a^{\frac{1}{4}} + b^{\frac{1}{4}}$ is $a^{\frac{3}{4}} - a^{\frac{1}{2}}b^{\frac{1}{4}} + a^{\frac{1}{4}}b^{\frac{1}{2}} - b^{\frac{3}{4}}$.
17. Of $\sqrt[3]{a} + \sqrt[3]{b}$ is $\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}$.

**247. Dividing polynomials involving positive and negative exponents.**

Divide $x^{\frac{2}{3}} - x^{\frac{1}{3}} - 6$ by $x^{\frac{1}{3}} - 3$.

$$\begin{array}{l|l} x^{\frac{2}{3}} - x^{\frac{1}{3}} - 6 & x^{\frac{1}{3}} - 3 \\ \hline x^{\frac{2}{3}} - 3x^{\frac{1}{3}} & x^{\frac{1}{3}} + 2 \\ \hline +2x^{\frac{1}{3}} - 6 & \\ 2x^{\frac{1}{3}} - 6 & \\ \hline \end{array}$$

Divide:

1. $x - y$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.
2. $a - b$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.
3. $x + y$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.
4. $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.
5. $a - c$ by $a^{\frac{1}{4}} - c^{\frac{1}{4}}$.
6. $a + b$ by $a^{\frac{1}{5}} + b^{\frac{1}{5}}$.
7. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.
8. $x - y$ by $x^{\frac{1}{4}} + y^{\frac{1}{4}}$.
9. $a + b$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.
10. $x^{\frac{4}{3}} + x^{\frac{2}{3}}a^{\frac{2}{3}} + a^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}$.
11. $2x^5y^{-3} - 5x^4y^{-2} + 7x^3y^{-1} - 5x^2 + 2xy$ by $x^3y^{-3} - x^2y^{-2} + xy^{-1}$.

**248. Extracting square root.**

Extract square root of $x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

$$\begin{array}{l|l} x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y & x^{\frac{1}{2}} - y^{\frac{1}{2}} \\ \hline x & \\ \hline 2x^{\frac{1}{2}} - y^{\frac{1}{2}} & -2x^{\frac{1}{2}}y^{\frac{1}{2}} + y \\ & -2x^{\frac{1}{2}}y^{\frac{1}{2}} + y \\ \hline \end{array}$$

Find the square root of:

1. $9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2$.
2. $4a - 12a^{\frac{1}{3}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}} + 16a^{\frac{1}{3}}c^{\frac{1}{3}} - 24b^{\frac{1}{3}}c^{\frac{1}{3}} + 16c^{\frac{1}{2}}$.

### GEOMETRIC FORMULAS INVOLVING POWERS AND ROOTS

**249.** The Greek letter $\pi$, called *pi*, always stands for an irrational number whose approximate value is 3.1416.

If $R$ = the radius, and $A$ = the area, of a circle, then

$$A = \pi R^2. \qquad (1)$$

1. Interpret this formula.

2. Solve the formula for $R$, and give its interpretation.

If $R$ = the radius, $S$ = the surface, and $V$ = the volume of a sphere, then

$$S = 4\pi R^2. \qquad (2) \qquad V = \tfrac{4}{3}\pi R^3. \qquad (3)$$

3. Interpret formula (2). Interpret formula (3).

4. Solve (2) for $R$, and give the interpretation.

5. Solve (3) for $R$, and give the interpretation.

If $H$ = the altitude, $R$ = the radius of the base, and $V$ = the volume of a circular cylinder, then

$$V = H \times \pi R^2 = \pi H R^2. \qquad (4)$$

6. Interpret this formula.

7. Solve (4) for $H$, and give the interpretation.

8. Solve (4) for $R$, and give the interpretation.

9. Find the radius of a circle whose area is 78.54 sq. yd.

10. Find the radius of a sphere whose surface is 28.2744 sq. in.

11. Find the radius of a sphere whose volume is 65.45 cu. in.

12. The altitude of a cylindrical cistern is 11 ft., and it holds 4136.14$^+$ gal. water. Find the diameter of its base (231 cu. in. in a gal.).

# XII. RADICALS AND SURDS

## DEFINITIONS

**250.** A **radical** is an indicated root; a **radicand** is the number whose root is indicated; and a **surd** is an irrational having a rational radicand.

Thus, $\sqrt{16}$, $\sqrt[3]{5}$, $6\sqrt[4]{30}$, $\sqrt{-2}$, $\sqrt[5]{5+2\sqrt{3}}$ are radicals whose radicands are 16, 5, 30, $-2$, $5+2\sqrt{3}$; of these radicals $\sqrt[3]{5}$ and $6\sqrt[4]{30}$ alone are surds. $\sqrt{-2}$ is an imaginary number (§ 267).

**251.** The **coefficient** of a radical is the rational factor that multiplies it; and the **order** of the radical is determined by the index of the root. Two radicals are of the same order when they have the same root index.

Thus, the surds $5\sqrt[6]{3a}$ and $n\sqrt[6]{7b}$ are of the same order, viz., the 6th; and their coefficients are 5 and $n$, respectively.

**252.** A **quadratic surd** is one of the second order, and a **cubic surd** is one of the third order.

**253.** **Similar** or **like radicals** are radicals having the same order and the same radicand; otherwise they are dissimilar or unlike.

Thus, $7\sqrt[3]{5a}$ and $(3+2x)\sqrt[3]{5a}$ are similar radicals.

**254.** A surd is in its **simplest form** when the radicand is an integer and as small as possible.

## REDUCTION OF RADICALS

**255.** To **reduce** a radical is to change its form without changing its value.

In a radical of the form $\sqrt[r]{a^n}$, $r$ is called the index and $n$ the exponent.

**256. Reducing radicals.**

(*a*) When the index and exponent have a common factor.

Simplify $\sqrt[6]{a^2}$ and $\sqrt[8]{64a^{12}b^6}$.

$\sqrt[6]{a^2} = a^{\frac{2}{6}} = a^{\frac{1}{3}} = \sqrt[3]{a}$.

$\sqrt[8]{64\,a^{12}b^6} = \sqrt[8]{(2\,a^2b)^6} = (2\,a^2b)^{\frac{6}{8}} = (2\,a^2b)^{\frac{3}{4}} = \sqrt[4]{(2\,a^2b)^3}$.

*Cancel the common factor in the index and exponent.*

Simplify:

1. $\sqrt[4]{a^2}$.
2. $\sqrt[6]{(ab)^3}$.
3. $\sqrt[6]{27\,c^6y^3}$.
4. $\sqrt[8]{a^6}$.
5. $\sqrt[4]{a^2b^2}$.
6. $\sqrt[8]{81\,a^4c^{12}}$.
7. $\sqrt[4]{25}$.
8. $\sqrt[9]{c^6x^6}$.
9. $\sqrt[9]{64\,x^3y^6}$.
10. $\sqrt[6]{16}$.
11. $\sqrt[8]{a^6x^4}$.
12. $\sqrt[10]{a^5b^{15}c^{10}}$.
13. $\sqrt[4]{\frac{36}{49}}$.
14. $\sqrt[6]{\frac{n^3}{8\,a^9}}$.
15. $\sqrt[8]{\frac{16\,a^4}{(x+2)^4}}$.

(*b*) When the exponent of a factor of the radicand is the same as the index.

Simplify $\sqrt[4]{a^4b}$, $\sqrt[3]{128\,a^4b}$, and $5\,a\sqrt{32\,a^5c^2}$.

$\sqrt[4]{a^4b} = \sqrt[4]{a^4} \times \sqrt[4]{b} = a\sqrt[4]{b}$.

$\sqrt[3]{128\,a^4b} = \sqrt[3]{64\,a^3} \times \sqrt[3]{2\,ab} = \sqrt[3]{64\,a^3} \times \sqrt[3]{2\,ab} = 4\,a\sqrt[3]{2\,ab}$.

$5\,a\sqrt{32\,a^5c^2} = 5\,a\sqrt{16\,a^4c^2 \times 2\,a} = 5\,a\sqrt{16\,a^4c^2} \times \sqrt{2\,a} = 5a \times 4\,a^2c \times \sqrt{2\,a} = 20\,a^2c\sqrt{2\,a}$.

*Resolve the radicand into two factors such that one of them will be a perfect power having an exponent equal to the index.*

*Remove this factor from under the radical sign, extract its required root, and multiply the coefficient of the surd by the root obtained.*

Simplify:

1. $\sqrt{8}$.
2. $3\sqrt{20}$.
3. $\sqrt{25\,a^3b}$.
4. $\sqrt{27}$.
5. $2\sqrt[3]{16}$.
6. $3\sqrt{a^4b^3}$.
7. $\sqrt{12}$.
8. $3\sqrt[4]{80}$.
9. $5\sqrt{8\,a^3b^2}$.
10. $\sqrt{75}$.
11. $4\sqrt{180}$.
12. $3\sqrt{75\,x^4y}$.
13. $\sqrt[3]{40}$.
14. $5\sqrt[3]{500}$.
15. $2\sqrt[3]{81\,a^4b}$.
16. $\sqrt[3]{54}$.
17. $6\sqrt[4]{512}$.
18. $\sqrt[3]{160\,x^5y^3}$.
19. $\sqrt[3]{250}$.
20. $2\sqrt[6]{320}$.
21. $5\sqrt[3]{108\,x^6z}$.
22. $\sqrt[5]{224}$.
23. $\sqrt[4]{1250}$.
24. $3\sqrt[4]{80\,a^4c}$.
25. $\sqrt[3]{432\,a^4b^3}$.
26. $4\,a\sqrt[5]{64\,a^{10}b^5c}$.
27. $7\sqrt[2]{625\,c^5x^3}$.
28. $5\,x\sqrt[3]{432\,c^3x^4y}$.
29. $2\sqrt{98\,a^5b^4x^3}$.
30. $\sqrt[3]{16\,x^3y^4 - 40\,x^4y^3}$.
31. $4\sqrt[3]{200\,x^6y^3}$.
32. $\sqrt{ax^2 - 6\,ax + 9\,a}$.
33. $5\sqrt[4]{648\,a^8b^2}$.
34. $\sqrt{(x^2 - y^2)(x + y)}$.
35. $3\cdot a^2b\sqrt[3]{40\,a^4}$.
36. $\sqrt[3]{192\,a^4b^5 + 320\,a^3b^4}$.

(*c*) When the radicand is a fraction.

Simplify $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{3}{8}}$, and $\sqrt[3]{\frac{2}{3}}$.

$$\sqrt{\tfrac{3}{4}} = \sqrt{\tfrac{1}{4} \times 3} = \sqrt{\tfrac{1}{4}} \times \sqrt{3} = \tfrac{1}{2}\sqrt{3}.$$

$$\sqrt{\frac{3}{8}} = \sqrt{\frac{3 \times 2}{8 \times 2}} = \sqrt{\frac{1}{16} \times 6} = \frac{1}{4}\sqrt{6}.$$

$$\sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{2 \times 9}{3 \times 9}} = \sqrt[3]{\frac{1}{27} \times 18} = \frac{1}{3}\sqrt[3]{18}.$$

*Multiply both terms of the fraction by the least number that will make the denominator a perfect power of the same degree as the radical; and then proceed as in the preceding case.*

Simplify:

1. $\sqrt{\frac{5}{9}}$.
2. $\sqrt{\frac{7}{18}}$.
3. $3\sqrt{\frac{2}{3}}$.
4. $\frac{1}{2}\sqrt{\frac{3}{7}}$.
5. $\sqrt{\frac{3}{8}}$.
6. $\sqrt{\frac{11}{15}}$.
7. $4\sqrt{1\frac{1}{2}}$.
8. $\frac{3}{4}\sqrt{\frac{2}{3}}$.
9. $\sqrt{\frac{5}{6}}$.
10. $\sqrt{\frac{17}{50}}$.
11. $6\sqrt{3\frac{1}{3}}$.
12. $\frac{5}{6}\sqrt{1\frac{2}{5}}$.
13. $\sqrt[3]{\frac{2}{5}}$.
14. $\sqrt[3]{\frac{3}{10}}$.
15. $8\sqrt[3]{\frac{3}{4}}$.
16. $1\frac{1}{4}\sqrt[3]{\frac{4}{7}}$.
17. $\sqrt[3]{\frac{1}{2}}$.
18. $\sqrt[3]{2\frac{1}{2}}$.
19. $5\sqrt[3]{2\frac{2}{5}}$.
20. $\frac{a}{b}\sqrt{\frac{b}{a}}$.
21. $\sqrt[3]{\frac{3}{5}}$.
22. $\sqrt[4]{1\frac{1}{3}}$.
23. $6\sqrt[4]{2\frac{5}{7}}$.
24. $\frac{a}{b}\sqrt[3]{\frac{b}{a}}$.
25. $\sqrt{\frac{a^2x}{b^3}}$.
26. $\sqrt[3]{\frac{c^2x}{b^2}}$.
27. $\frac{a}{b}\sqrt[4]{\frac{b^5c}{a^3}}$.
28. $a^3\left(1-\frac{b^2}{a^2}\right)^{\frac{3}{2}}$.
29. $x^6\left(1-\frac{1}{x^4}\right)^{\frac{3}{4}}$.

**257. Carrying the coefficient of a radical under the radical sign.**

Carry the coefficients of $5a\sqrt{2ab}$ and $\frac{2}{3}\sqrt[3]{\frac{81}{4}}$ under the radical sign.

$$5a\sqrt{2ab} = \sqrt{(5a)^2} \times \sqrt{2ab} = \sqrt{25a^2 \times 2ab} = \sqrt{50a^3b}.$$

$$\tfrac{2}{3}\sqrt[3]{\tfrac{81}{4}} = \sqrt[3]{(\tfrac{2}{3})^3} \times \sqrt[3]{\tfrac{81}{4}} = \sqrt[3]{\tfrac{8}{27} \times \tfrac{81}{4}} = \sqrt[3]{6}.$$

*Raise the coefficient to a power of the same degree as the radical, multiply this power by the given radicand, and indicate the required root of the product.*

NOTE. This process is simply the reverse of that of § 256.

Carry coefficients under the radical signs:

1. $2\sqrt{5}$.
2. $3\sqrt[4]{2}$.
3. $2a\sqrt{3ab}$.
4. $3\sqrt{2}$.
5. $n\sqrt[3]{a}$.
6. $a^2b\sqrt[3]{ab}$.
7. $4\sqrt{3}$.
8. $c\sqrt[4]{7}$.
9. $2a^2b\sqrt[4]{2ac}$.
10. $2\sqrt[3]{5}$.
11. $a^2\sqrt[5]{a}$.
12. $-3\sqrt{2x}$.
13. $3\sqrt[3]{2}$.
14. $c^2\sqrt[3]{c^3}$.
15. $-3\sqrt[3]{2x}$.
16. $\frac{1}{2}\sqrt{\frac{4}{5}}$.
17. $\frac{3}{4}\sqrt[3]{\frac{16}{81}}$.
18. $\frac{a}{b}\sqrt{\frac{b}{a}}$.
19. $\frac{2}{3}\sqrt{\frac{9}{8}}$.
20. $4\sqrt[3]{\frac{3}{8}}$.
21. $\frac{a}{b}\sqrt[3]{\frac{b}{a}}$.
22. $3a^2b\sqrt{5abc}$.
23. $2ax\sqrt{a^2+3ax}$.
24. $\frac{x+y}{x-y}\sqrt{\frac{x-y}{x+y}}$.
25. $(x-1)\sqrt{\frac{2}{x-1}+1}$.
26. $(x-y)\sqrt{\frac{1}{x^2-y^2}}$.
27. $(a-b)\sqrt[3]{\frac{1}{(a-b)^2}}$.

**258. Reducing radicals to a common index.**

Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to a common index.

The L. C. M. of the indexes 2, 3, 4 is 12, which is to be the common index.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64}.$$

$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{3^4} = \sqrt[12]{81}.$$

$$\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{5^3} = \sqrt[12]{125}.$$

*Take the L. C. M. of the indexes for the common index; and raise each radicand to the power whose exponent is the quotient of the L. C. M. by the index of the radical.*

Reduce to a common index:

1. $\sqrt{5}$ and $\sqrt[3]{11}$.
2. $\sqrt[3]{2}$, $\sqrt[4]{3}$, and $\sqrt[6]{5}$.
3. $\sqrt[3]{2}$ and $\sqrt[5]{3}$.
4. $\sqrt{2}$, $\sqrt[5]{3}$, and $\sqrt[10]{5}$.
5. $\sqrt[4]{4}$ and $\sqrt[5]{5}$.
6. $\sqrt[9]{3}$, $\sqrt[3]{2}$, and $\sqrt[6]{7}$.
7. Which is the greater, $\sqrt{2}$ or $\sqrt[3]{3}$?

$\sqrt{2} = \sqrt[6]{8}$ and $\sqrt[3]{3} = \sqrt[6]{9}$. $\therefore$ $\sqrt[3]{3}$ is greater than $\sqrt{2}$.

Find which is the greater:

8. $\sqrt[3]{2}$ or $\sqrt[4]{3}$.
9. $\sqrt{7}$ or $\sqrt[3]{18}$.
10. $\sqrt[3]{5}$ or $\sqrt[5]{14}$.
11. $\sqrt{3}$ or $\sqrt[5]{16}$.
12. $\sqrt[4]{x^7}$ or $\sqrt[5]{x^9}$.
13. $\sqrt[6]{a^7}$ or $\sqrt[7]{a^8}$.
14. $\sqrt{10}$ or $\sqrt[3]{32}$.
15. $\sqrt[3]{7}$ or $\sqrt[4]{15}$.
16. $\sqrt[4]{11}$ or $\sqrt[6]{36}$.
17. $5\sqrt{6}$ or $6\sqrt{5}$.
18. $6\sqrt{11}$ or $5\sqrt{15\frac{4}{5}}$.
19. $\sqrt{\frac{2}{5}}$ or $\sqrt[3]{\frac{14}{15}}$.

**259. Adding and subtracting radicals.**

1. Find $3\sqrt{5}+8\sqrt{5}-4\sqrt{5}$.

By the distributive law (§ 33 and § 89), we have

$$3\sqrt{5}+8\sqrt{5}-4\sqrt{5}=(3+8-4)\sqrt{5}=7\sqrt{5}.$$

2. Find $\sqrt{18}+\sqrt{72}-\sqrt{50}$.

$$\begin{aligned}
+\sqrt{18} &= +\sqrt{9\times 2} &&= +3\sqrt{2}\\
+\sqrt{72} &= +\sqrt{36\times 2} &&= +6\sqrt{2}\\
-\sqrt{50} &= -\sqrt{25\times 2} &&= -5\sqrt{2}\\
\text{Sum} & &&= +4\sqrt{2}
\end{aligned}$$

3. Find $\sqrt{\frac{1}{6}}-\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}$.

$$\begin{aligned}
+\sqrt{\tfrac{1}{6}} &= +\sqrt{\tfrac{6}{36}} &&= +\tfrac{1}{6}\sqrt{6}\\
-\sqrt{\tfrac{2}{3}} &= -\sqrt{\tfrac{8}{9}} &&= -\tfrac{1}{3}\sqrt{6}\\
+\sqrt{\tfrac{3}{2}} &= +\sqrt{\tfrac{9}{4}} &&= +\tfrac{1}{2}\sqrt{6}\\
\text{Sum} & &&= +\tfrac{1}{3}\sqrt{6}
\end{aligned}$$

4. Find $\sqrt{20}+\sqrt{63}+\sqrt{125}-\sqrt{112}$.

$$\begin{aligned}
+\sqrt{20} &= +2\sqrt{5}\\
+\sqrt{63} &= +3\sqrt{7}\\
+\sqrt{125} &= +5\sqrt{5}\\
-\sqrt{112} &= -4\sqrt{7}\\
\text{Sum} &= 7\sqrt{5}-\sqrt{7}
\end{aligned}$$

*To add or subtract surds, reduce each surd to its simplest form.*

*If the surds are similar, add their coefficients, and to the sum annex the common surd factor.*

*If the surds are dissimilar, connect them with their proper signs.*

Find:

1. $\sqrt{50}+\sqrt{8}$.
2. $3\sqrt{12}+2\sqrt{75}-4\sqrt{48}$.
3. $\sqrt{27}-\sqrt{12}$.
4. $4\sqrt{18}-3\sqrt{50}+5\sqrt{98}$.
5. $\sqrt{a^2c}+a\sqrt{9c}$.
6. $\sqrt[3]{320}-\sqrt[3]{135}+3\sqrt[3]{40}$.
7. $\sqrt[3]{24}+\sqrt[3]{54}$.
8. $5\sqrt[4]{3}-3\sqrt[4]{48}+\sqrt[4]{243}$.
9. $\sqrt[3]{a^3x}+a\sqrt[3]{8x}$.
10. $2\sqrt{98}-3\sqrt{63}+5\sqrt{28}$.
11. $2\sqrt{18\,a}-\sqrt{8\,a}$.
12. $\sqrt{32\,a^3b^2}+\sqrt{18\,a^3b^2}$.
13. $\sqrt[4]{80}-\sqrt[4]{5\,a^4}$.
14. $2\sqrt[3]{81\,a^6c}+5\sqrt[3]{192\,a^6c}$.
15. $\sqrt{363\,a^5x^6y}-\sqrt{243\,a^5x^6y}+2\,a^2x^3\sqrt{3\,ay}$.
16. $\sqrt[3]{256\,a^6x^4}+a\sqrt[3]{108\,a^3x^4}-3\,a^2x\sqrt[3]{32\,x}$.

---

17. $\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{6}}$.
18. $\frac{1}{2}\sqrt{\frac{3}{8}}+\frac{3}{4}\sqrt{\frac{1}{6}}$.
19. $\sqrt{\frac{1}{8}}+\sqrt{\frac{1}{2}}$.
20. $\frac{2}{3}\sqrt[3]{\frac{9}{4}}-\frac{3}{4}\sqrt[3]{\frac{2}{3}}$.
21. $\sqrt{\frac{3}{4}}-\sqrt{\frac{1}{3}}$.
22. $5\sqrt{\frac{a}{b}}-a\sqrt{\frac{1}{ab}}$.

---

23. $50^{\frac{1}{2}}+8^{\frac{1}{2}}-98^{\frac{1}{2}}$.
24. $250^{\frac{1}{3}}-16^{\frac{1}{3}}-54^{\frac{1}{3}}$.
25. $(\frac{4}{3})^{\frac{1}{2}}+(\frac{25}{12})^{\frac{1}{2}}-(\frac{1}{3})^{\frac{1}{2}}$.
26. $(\frac{8}{5})^{\frac{1}{3}}+(\frac{1}{5})^{\frac{1}{3}}+(5\frac{2}{5})^{\frac{1}{3}}$.
27. $2(3)^{\frac{1}{2}}-\frac{1}{2}(12)^{\frac{1}{2}}+4(27)^{\frac{1}{2}}$.
28. $(40)^{\frac{1}{3}}+(25)^{\frac{1}{3}}+2(625)^{\frac{1}{3}}$.
29. $(\frac{5}{3})^{\frac{1}{2}}+(\frac{3}{5})^{\frac{1}{2}}+(\frac{1}{15})^{\frac{1}{2}}$.
30. $(5\frac{1}{3})^{\frac{1}{3}}-(27)^{\frac{1}{3}}+(208\frac{1}{3})^{\frac{1}{3}}$.

**260. Multiplying and dividing radicals.**

1. Find $5\sqrt{6} \times 8\sqrt{10}$.

$5\sqrt{6} \times 8\sqrt{10} = 5 \times 8 \times \sqrt{6} \times \sqrt{10} = 40\sqrt{60} = 80\sqrt{15}$.

2. Find $4\sqrt{2} \times 5\sqrt[3]{3}$.

$4\sqrt{2} = 4\sqrt[6]{2^3}$, $5\sqrt[3]{3} = 5\sqrt[6]{3^2}$.

$\therefore 4\sqrt{2} \times 5\sqrt[3]{3} = 4\sqrt[6]{8} \times 5\sqrt[6]{9} = 20\sqrt[6]{72}$.

3. Find $3\sqrt{6} \div 2\sqrt{5}$.

$3\sqrt{6} \div 2\sqrt{5} = \frac{3}{2}\sqrt{\frac{6}{5}} = \frac{3}{10}\sqrt{30}$.

*Reduce the radicals to a common index.*

*Find the product (or quotient) of the coefficients for the required coefficient, and the product (or quotient) of the radicands for the required radicand.*

*Reduce the result to its simplest form.*

Find :

1. $\sqrt{3} \times \sqrt{6}$.
2. $3\sqrt{5} \times 2\sqrt{10}$.
3. $4\sqrt[3]{16} \div 2\sqrt[3]{2}$.
4. $\sqrt{2} \times \sqrt{8}$.
5. $6\sqrt{8} \div 3\sqrt{2}$.
6. $5\sqrt[3]{32} \times 3\sqrt[3]{2}$.
7. $\sqrt[3]{3} \times \sqrt[3]{9}$.
8. $8\sqrt{6} \div 4\sqrt{3}$.
9. $10\sqrt[3]{2} \div 2\sqrt[3]{5}$.
10. $\sqrt[3]{2} \times \sqrt[3]{4}$.
11. $6\sqrt{5} \div 3\sqrt{2}$.
12. $12\sqrt[4]{2} \div 4\sqrt[4]{3}$.
13. $\sqrt{2} \times \sqrt[6]{3}$.
14. $6\sqrt[3]{2} \div 2\sqrt[6]{3}$.
15. $10\sqrt[3]{2} \times 2\sqrt[4]{3}$.
16. $4\sqrt{6a} \times 3\sqrt{8ac}$.
17. $7\sqrt{10a^2} \times 2\sqrt{5ab^2}$.
18. $9\sqrt{54a^2} \div 3\sqrt{3}$.
19. $5\sqrt{14a^5} \times 2x\sqrt{7a^2c}$.
20. $15\sqrt{6a^3x} \div 5\sqrt{2a}$.
21. $\sqrt{162a^5b^2} \div \sqrt{8a^3b^4}$.

Find:

22. $\frac{1}{4}\sqrt{6} \times \frac{3}{14}\sqrt{8}$.

23. $\frac{2}{3}\sqrt{\frac{3}{4}} \div \frac{1}{2}\sqrt{\frac{4}{3}}$.

24. $4\sqrt[3]{\frac{2}{3}} \times 3\sqrt[3]{\frac{5}{6}}$.

25. $\frac{1}{2}\sqrt[3]{\frac{2}{3}} \div \frac{1}{3}\sqrt[3]{\frac{1}{2}}$.

26. $3\sqrt[3]{4} \times 4\sqrt{3}$.

27. $2\sqrt[3]{3} \div 3\sqrt{2}$.

NOTE. The $n$th power of the $n$th root of a number is equal to that number.

Thus, $(\sqrt{7})^2 = 7$, $(\sqrt[3]{1+x+x^2})^3 = 1+x+x^2$.

28. Multiply $\sqrt{2}+4\sqrt{3}$ by $\sqrt{2}-7\sqrt{3}$.

$$\begin{array}{r} \sqrt{2}+4\sqrt{3} \\ \sqrt{2}-7\sqrt{3} \\ \hline 2+4\sqrt{6} \\ -7\sqrt{6}-28\times 3 \\ \hline 2-3\sqrt{6}-84 \end{array} \qquad = -82-3\sqrt{6}.$$

Multiply:

29. $\sqrt{7}+\sqrt{5}$ by $\sqrt{7}-\sqrt{5}$.

30. $\sqrt{6}+\sqrt{8}$ by $\sqrt{6}+\sqrt{8}$.

31. $2\sqrt{3}-5\sqrt{2}$ by $3\sqrt{3}+4\sqrt{2}$.

32. $5+3\sqrt{7}$ by $9-4\sqrt{7}$.

33. $\sqrt{2}+\sqrt{3}+\sqrt{5}$ by $\sqrt{2}-\sqrt{3}+\sqrt{5}$.

34. $3\sqrt{5}-2\sqrt{6}+\sqrt{7}$ by $6\sqrt{5}+4\sqrt{6}+2\sqrt{7}$.

35. $\sqrt{a}+\sqrt{b}+\sqrt{c}$ by $\sqrt{a}-\sqrt{b}+\sqrt{c}$.

36. $\sqrt{1+x}+2\sqrt{1-x}$ by $\sqrt{1+x}-5\sqrt{1-x}$.

37. $\sqrt{2+3x}+7\sqrt{3+2x}$ by $3\sqrt{2+3x}-4\sqrt{3+2x}$.

38. $\sqrt{1+x}+2\sqrt{1+2x}-3\sqrt{1-3x}$ by $\sqrt{1+x}-2\sqrt{1+2x}+3\sqrt{1-3x}$.

**261. Rationalizing the denominator of a fraction.**

To find the approximate numerical value of $\frac{5}{\sqrt{2}}$, as it stands, requires two tedious operations:

(*a*) Finding the approximate value of $\sqrt{2}$, $1.4142\cdots$.

(*b*) Dividing 5 by $1.4142\cdots$.

Multiplying both terms of the fraction by $\sqrt{2}$, we get $\frac{5\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{5\sqrt{2}}{2}$, a form in which the required value is much more easily determined.

In general, a fraction having one or more radicals in its denominator is simplified by rationalizing the denominator.

**262.** A **rationalizing factor** is the factor by which a given surd (or radical) must be multiplied in order to obtain a rational product.

Thus, of the two surds $\sqrt[3]{2}$ and $\sqrt[3]{4}$, each is the rationalizing factor of the other, since their product, 2, is rational.

Of two binomial quadratic surds like $\sqrt{m}+\sqrt{n}$ and $\sqrt{m}-\sqrt{n}$, which differ only in the sign of one term, each is called the **conjugate** of the other. Moreover, since the product of any two conjugate quadratic surds is rational (§ 81), each of them is the rationalizing factor of the other.

1. Simplify $\frac{2+x}{\sqrt[3]{5}}$.

Multiply both terms by $\sqrt[3]{25}$. The result is $\frac{(2+x)\sqrt[3]{25}}{5}$.

2. Simplify $\frac{5+\sqrt{3}}{5-\sqrt{3}}$.

Multiply both terms by the conjugate of the denominator, $5+\sqrt{3}$. The result is $\frac{28+10\sqrt{3}}{22}=\frac{14+5\sqrt{3}}{11}$.

Simplify:

1. $\frac{3}{\sqrt{2}}$.
2. $\frac{5}{\sqrt{8}}$.
3. $\frac{2+\sqrt{3}}{\sqrt{3}}$.
4. $\frac{1}{\sqrt[3]{4}}$.
5. $\frac{6}{\sqrt[3]{9}}$.
6. $\frac{\sqrt[3]{4}-\sqrt[3]{5}}{2\sqrt[3]{4}}$.
7. $\frac{5}{\sqrt[4]{5}}$.
8. $\frac{2}{\sqrt[3]{2}}$.
9. $\frac{2+2\sqrt[4]{5}}{\sqrt[4]{8}}$.
10. $\frac{4-\sqrt{3}}{4+\sqrt{3}}$.
11. $\frac{3+\sqrt{5}}{3-\sqrt{5}}$.
12. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.
13. $\frac{2\sqrt{5}+\sqrt{6}}{2\sqrt{5}-\sqrt{6}}$.
14. $\frac{8-3\sqrt{2}}{8+3\sqrt{2}}$.
15. $\frac{4\sqrt{5}+2\sqrt{3}}{4\sqrt{5}-2\sqrt{3}}$.
16. $\frac{2+\sqrt{a+1}}{2-\sqrt{a+1}}$.
17. $\frac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.
18. $\frac{x+\sqrt{x^2-4}}{x-\sqrt{x^2-4}}$.
19. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.
20. $\frac{\sqrt{a^2-1}-\sqrt{a^2+1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}$.

**263. Finding powers and roots of radicals.**

A power or root of a radical is easily found by using fractional exponents.

1. Find the square of $3\sqrt[3]{a}$.

$(3\sqrt[3]{a})^2=(3a^{\frac{1}{3}})^2=3^2a^{\frac{2}{3}}=9\sqrt[3]{a^2}$.

2. Find the fourth power of $2\sqrt[5]{a^2b}$.

$(2\sqrt[5]{a^2b})^4=(2a^{\frac{2}{5}}b^{\frac{1}{5}})^4=2^4a^{\frac{8}{5}}b^{\frac{4}{5}}=16a\sqrt[5]{a^3b^4}$.

3. Find the cube root of $9a\sqrt{bc^3}$.

$(9a\sqrt{bc^3})^{\frac{1}{3}}=(9ab^{\frac{1}{2}}c^{\frac{3}{2}})^{\frac{1}{3}}=9^{\frac{1}{3}}a^{\frac{1}{3}}b^{\frac{1}{6}}c^{\frac{1}{2}}=9^{\frac{2}{6}}a^{\frac{2}{6}}b^{\frac{1}{6}}c^{\frac{3}{6}}=\sqrt[6]{9^2a^2bc^3}=\sqrt[6]{81a^2bc^3}$.

Find the:

1. 4th power of $2\sqrt{a}$.
2. 4th power of $2x^2\sqrt[3]{6}$.
3. 2d power of $3\sqrt[4]{c}$.
4. 9th power of $-x\sqrt[6]{a^5}$.
5. 6th power of $\frac{2}{3}\sqrt{3}$.
6. 3d power of $-3\sqrt[4]{a^2x}$.
7. 3d power of $5a\sqrt[3]{4}$.
8. 8th power of $2\sqrt[6]{a^3b^2}$.

Find the:

9. Cube root of $8\sqrt{a^3}$.
10. Square root of $5\sqrt[3]{16}$.
11. Fourth root of $9\sqrt[3]{a^2}$.
12. Tenth root of $a^5\sqrt[7]{b^{20}}$.
13. Sixth root of $x^{12}\sqrt{a^5}$.
14. Cube root of $27\sqrt[3]{8a^6}$.
15. Cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$.
16. Fourth root of $\frac{4}{9}\sqrt[3]{\frac{4}{9}}$.

Perform the following indicated operations:

17. $(a\sqrt{a})^2$.
18. $\sqrt{\sqrt{a^3}}$.
19. $\sqrt[2]{9\sqrt[2]{36}}$.
20. $(c\sqrt[3]{c})^3$.
21. $\sqrt{\sqrt[3]{a^4}}$.
22. $\sqrt[3]{8\sqrt{27}}$.
23. $(\sqrt{a^3c})^4$.
24. $\sqrt[3]{\sqrt{a^9}}$.
25. $\sqrt[4]{a^4\sqrt[3]{a^{16}}}$.
26. $(\sqrt[3]{b^5})^{12}$.
27. $\sqrt[4]{\sqrt[3]{a^8}}$.
28. $\sqrt[3]{c^9\sqrt[5]{c^{12}}}$.

Show by fractional exponents that:

29. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}}$.
30. $\sqrt[m]{a^n} = (\sqrt[m]{a})^n$.
31. Find, by formula 29, $\sqrt[4]{\sqrt[3]{256}}$, $\sqrt[5]{\sqrt[4]{32\,a^5}}$.
32. Find, by formula 30, $\sqrt[3]{64^2}$, $\sqrt[4]{625^3}$.

**264. Extracting the square roots of binomial surds.**

1. Find the square root of $8 + 4\sqrt{3}$.

The square root of a number of this kind can often be found by inspection.

Thus, since $(\sqrt{a} + \sqrt{b})^2 = (a + b) + 2\sqrt{ab}$, the square root of $(a + b) + 2\sqrt{ab} = \sqrt{a} + \sqrt{b}$.

Hence, if the given number is written under the form of $m + 2\sqrt{n}$, we have only to find two numbers whose sum is $m$ and whose product is $n$, and connect their square roots by the sign of the second term of the surd.

Thus, $8 + 4\sqrt{3} = 8 + 2 \times 2\sqrt{3} = 8 + 2\sqrt{12}$.

Now, we find by inspection that $6 + 2 = 8$ and $6 \times 2 = 12$. Hence, the square root of $8 + 4\sqrt{3}$ is $\sqrt{6} + \sqrt{2}$.

2. Find the square root of $12 - 8\sqrt{2}$.

$12 - 8\sqrt{2} = 12 - 2\sqrt{32}$. $8 + 4 = 12$ and $8 \times 4 = 32$.

Hence, $\sqrt{12 - 8\sqrt{2}} = \sqrt{4} - \sqrt{8} = 2 - 2\sqrt{2}$.

3. Find the square root of $15 + \sqrt{216}$.

$15 + \sqrt{216} = 15 + 2\sqrt{54}$; $9 + 6 = 15$, $9 \times 6 = 54$.

Hence, $\sqrt{15 + \sqrt{216}} = \sqrt{9} + \sqrt{6} = 3 + \sqrt{6}$.

Find the square root of :

| | | |
|---|---|---|
| 1. $4 + 2\sqrt{3}$. | 2. $7 - 4\sqrt{3}$. | 3. $7 + 2\sqrt{10}$. |
| 4. $18 + 8\sqrt{5}$. | 5. $17 + 4\sqrt{15}$. | 6. $13 - 2\sqrt{30}$. |
| 7. $10 + 2\sqrt{21}$. | 8. $11 - 6\sqrt{2}$. | 9. $38 - 12\sqrt{10}$. |
| 10. $16 - 2\sqrt{55}$. | 11. $35 + 10\sqrt{10}$. | 12. $24 - \sqrt{252}$. |

Find in two ways the square root of :

| | | |
|---|---|---|
| 13. $13 + 2\sqrt{36}$. | 14. $29 + 2\sqrt{100}$. | 15. $25 + 2\sqrt{144}$. |
| 16. $26 + 2\sqrt{25}$. | 17. $40 + 2\sqrt{144}$. | 18. $53 + 2\sqrt{196}$. |

### AN IMPORTANT PROPERTY OF QUADRATIC SURDS

**265.** *Neither the sum nor the difference of two dissimilar quadratic surds can be a rational number.*

PROOF. Let $\sqrt{a}$ and $\sqrt{b}$ be two dissimilar surds.

If possible let $\sqrt{a} \pm \sqrt{b} = r$, (1)

$r$ being a rational number, and not zero.

From (1), $\pm\sqrt{b} = r - \sqrt{a}$. (2)

Square (2), $b = r^2 - 2r\sqrt{a} + a$. (3)

Solve (3) for $\sqrt{a}$, $\sqrt{a} = \dfrac{r^2 + a - b}{2r}$.

Hence, if (1) were true, then the surd $\sqrt{a}$ would be equal to the rational number $\dfrac{r^2 + a - b}{2r}$, which is impossible. Hence, (1) cannot be true.

**266.** It follows from the above property that if

$$x + \sqrt{y} = a + \sqrt{b}, \tag{4}$$

where $x$ and $a$ are rational and $\sqrt{y}$ and $\sqrt{b}$ are irrational, then $x = a$ and $y = b$.

For, (4) may be written $\sqrt{y} - \sqrt{b} = a - x$, which, by § 265, can be true only if each member is zero; that is, if $x = a$ and $y = b$. That is, the equation $x + \sqrt{y} = a + \sqrt{b}$ is equivalent to the two equations $x = a$ and $y = b$.

This principle may be used in extracting the square root of a binomial quadratic surd.

Thus, let $\sqrt{x} + \sqrt{y} = \sqrt{9 + 2\sqrt{14}}$. (1)

Square (1), $x + y + 2\sqrt{xy} = 9 + 2\sqrt{14}$.

Hence, by the above principle,

$x + y = 9$ and $xy = 14$.

Whence, $x = 7$ or $2$, and $y = 2$, or $7$.

## COMPLEX NUMBERS

**267.** A **pure imaginary number** is the indicated even root of a negative number.

Thus, $\sqrt{-4}$, $3\sqrt[4]{-2}$, $-a\sqrt[6]{-b}$ are pure imaginaries.

Any expression containing a pure imaginary is also said to be imaginary.

A **complex number** is the sum of a real and a pure imaginary number.

Thus, $3+\sqrt{-2}$, $a+n\sqrt{-b}$ are complex numbers.

The **imaginary unit** is $\sqrt{-1}$, which is often denoted by $i$.

Therefore $i=\sqrt{-1}$; $i^2=(\sqrt{-1})^2=-1$; $i^3=i^2\times i=-1\times\sqrt{-1}=-\sqrt{-1}$; $i^4=i^2\times i^2=-1\times-1=+1$; $i^5=i^4\times i=+1\times\sqrt{-1}=\sqrt{-1}=i$, $i^6=i^2$; $i^7=i^3$, etc.

Hence, the successive powers of $i=\sqrt{-1}$ form the repeating series $+\sqrt{-1}$, $-1$, $-\sqrt{-1}$, $+1$.

In operations with imaginaries, while not necessary, it is better to express them in terms of $i$.

Thus, $\sqrt{-4}=\sqrt{4\times-1}=2\sqrt{-1}=2i$; $\sqrt{-3}=\sqrt{3\times-1}=\sqrt{3}\,i$; $4+2\sqrt{-18}=4+6\sqrt{2}\,i$, and so on.

1. Find the sum of $\sqrt{-9}$ and $\sqrt{-25}$.

$\sqrt{-9}+\sqrt{-25}=3i+5i=8i=8\sqrt{-1}$.

2. Find the product of $\sqrt{-3}$ by $\sqrt{-12}$.

$\sqrt{-3}\times\sqrt{-12}=\sqrt{3}\,i\times\sqrt{12}\,i=\sqrt{36}\,i^2=-6$.

3. Find the value of $\dfrac{3+\sqrt{-25}}{3-\sqrt{-25}}$ or $\dfrac{3+5i}{3-5i}$.

Multiply both terms by $3+5i$, § 262. The result is:

$$\frac{9+30i+25i^2}{9-25i^2}=\frac{9+30\sqrt{-1}-25}{9+25}=\frac{-16+30\sqrt{-1}}{34}=\frac{-8+15\sqrt{-1}}{17}.$$

Find the simplest value of:

1. $\sqrt{-81}+\sqrt{-49}$.
2. $6+3\sqrt{-4}+5+2\sqrt{-9}$.
3. $\sqrt{-64}-\sqrt{-25}$.
4. $9-2\sqrt{-8}-4+5\sqrt{-2}$.
5. $\sqrt{-16}\times\sqrt{-36}$.
6. $(3+2\sqrt{-1})\times(4+3\sqrt{-1})$.
7. $\sqrt{-100}\div\sqrt{-4}$.
8. $(5+3\sqrt{-1})\div(5-3\sqrt{-1})$.
9. $(2\sqrt{-4})^3$.
10. $(2+5\sqrt{-1})\times(2-5\sqrt{-1})$.
11. $(5\sqrt{-3})^2$.
12. $(3+7\sqrt{-2})\times(3-7\sqrt{-2})$.
13. $(3\sqrt{-2})^4$.
14. $(5+2\sqrt{-1})\div(5-2\sqrt{-1})$.
15. $(2\sqrt{-2})^5$.
16. $(7-3\sqrt{-2})\div(7+3\sqrt{-2})$.
17. $(1-3i)^2$.
18. $(5+4i)\times(3-2i)$.

Simplify:

19. $\dfrac{5+\sqrt{-7}}{5-\sqrt{-7}}$.
20. $\dfrac{\sqrt{-8}+4}{\sqrt{-8}-4}$.
21. $\dfrac{\sqrt{6}-3\sqrt{-3}}{\sqrt{6}+3\sqrt{-3}}$.

Verify the following:

22. One of the cube roots of 1 is $\dfrac{-1+\sqrt{3}\,i}{2}$.

HINT. See whether $\left(\dfrac{-1+\sqrt{3}\,i}{2}\right)^3$ is equal to 1.

23. One of the cube roots of 1 is $\dfrac{-1-\sqrt{3}\,i}{2}$.

24. One of the cube roots of $-1$ is $\dfrac{1+\sqrt{3}\,i}{2}$.

25. One of the cube roots of $-1$ is $\dfrac{1-\sqrt{3}\,i}{2}$.

26. One of the roots of $x^2-4x+5=0$ is $2+i$, and the other is $2-i$.

HINT. Substitute $2+i$ for $x$, etc.

27. One of the roots of $x^2+6x+12=0$ is $-3+\sqrt{3}\,i$, and the other is $-3-\sqrt{3}\,i$.

## SOLUTION OF RADICAL OR IRRATIONAL EQUATIONS

**268. Radical or irrational equations** are equations that contain indicated roots of the unknown number.

**269. When the equation is of the form $(ax)^p = b$, where $a$, $b$, and $p$ are any real numbers.**

1. Find the value of $x$ in $(2x)^{\frac{2}{3}} = 16$.

| | |
|---|---|
| Extract square root, | $(2x)^{\frac{1}{3}} = \pm 4.$ |
| Raise to 3d power, | $2x = \pm 64.$ |
| Divide by 2, | $x = \pm 32.$ *Ans.* |

2. Find the value of $x$ in $2x^{-2} = \frac{5}{8}$.

| | |
|---|---|
| Divide by 2, | $x^{-2} = \frac{5}{16}.$ |
| Take reciprocals, | $x^2 = \frac{16}{5}.$ |
| Extract square root, | $x = \pm \frac{4}{\sqrt{5}}$ |
| | $= \pm \frac{4}{5}\sqrt{5}.$ § 261. |

Solve the following:

| | | |
|---|---|---|
| 1. $\sqrt{3x} = 9.$ | 2. $2x^{\frac{1}{2}} = 10.$ | 3. $\sqrt[3]{3x^2} = 4.$ |
| 4. $3\sqrt{x} = 9.$ | 5. $(2x)^{\frac{1}{2}} = 10.$ | 6. $\sqrt[3]{(3x)^2} = 4.$ |
| 7. $\sqrt[3]{x^2} = 57.$ | 8. $(2x)^{\frac{2}{3}} = 9.$ | 9. $\sqrt[3]{12x} = 6.$ |
| 10. $\sqrt[4]{x^3} = 8.$ | 11. $2x^{\frac{2}{3}} = 9.$ | 12. $8x^{-3} = 27.$ |
| 13. $\sqrt[5]{x^2} = 4.$ | 14. $(3x)^{\frac{3}{4}} = 8.$ | 15. $9x^{-4} = 16.$ |
| 16. $\sqrt[5]{x^3} = 27.$ | 17. $3x^{\frac{3}{4}} = 8.$ | 18. $4x^{-\frac{2}{3}} = 9.$ |
| 19. $\sqrt[5]{x^4} = 16.$ | 20. $4x^{-\frac{2}{5}} = 1.$ | 21. $16x^{-\frac{4}{5}} = 1.$ |
| 22. $x^{\frac{1}{3}} = \frac{1}{2}$ or 2. | | 23. $x^{\frac{2}{3}} = 4$ or $-9.$ |
| 24. $x^{-\frac{1}{3}} = 1$ or $\frac{1}{8}.$ | | 25. $x^{-2} = 1$ or $\frac{1}{16}.$ |
| 26. $\sqrt[4]{x^3} = 1$ or 8. | | 27. $\sqrt{x^{-2}} = 1$ or $-\frac{2}{3}.$ |
| 28. $\sqrt[5]{x^{-2}} = 4$ or 2. | | 29. $(\frac{1}{2}x)^{-\frac{1}{6}} = 1$ or 32. |

**270. When one or more of the radicands is a polynomial.**

1. Solve the equation $\sqrt{x+3}+x=9$.

Transpose, $\sqrt{x+3}=9-x$.

Square both sides, $x+3=81-18x+x^2$.

Reduce, $x^2-19x+78=0$.

Whence, $x=13$ or $6$.

On substituting it is found that 13 satisfies the given equation if the negative value of $\sqrt{x+3}$ is taken, while 6 satisfies it if the positive value of $\sqrt{x+3}$ is taken.

2. Solve $\sqrt{x+1}+\sqrt{x-2}+\sqrt{4x-3}=0$.

Transpose, $\sqrt{x+1}+\sqrt{x-2}=-\sqrt{4x-3}$.

Square, $x+1+2\sqrt{x^2-x-2}+x-2=4x-3$.

Transpose, reduce, $2\sqrt{x^2-x-2}=2x-2$.

Divide by 2, $\sqrt{x^2-x-2}=x-1$.

Square, $x^2-x-2=x^2-2x+1$.

Whence, $x=3$.

On substituting it is found that the given equation is satisfied for $x=3$ if the negative value of $\sqrt{4x-3}$ is taken.

Note that the solution is the same as if the given equation were $\sqrt{x+1}+\sqrt{x-2}-\sqrt{4x-3}=0$.

If the equation contains one radical, as in Example 1, solve the equation for that radical, and then get rid of the radical sign by raising both sides to that power which is indicated by the index of the radical. If two radicals are involved, two steps of this kind may be necessary, as in Example 2.

Substitute the values found for $x$ in the given equation, and state what restrictions, if any, must be made on the signs of the radicals in order that the results shall be roots.

Solve:

1. $\sqrt{x^2+16}=x+2.$
2. $\sqrt{x^2+11}=x+1.$
3. $\sqrt{x+7}+\sqrt{x}=7.$
4. $\sqrt{x}+\sqrt{x+5}=5.$
5. $x-\sqrt{x^2-3x}=2.$
6. $6+\sqrt{x}=\sqrt{12+x}.$
7. $\sqrt{4x^2-19}=2x-1.$
8. $\sqrt{x-32}=16-\sqrt{x}.$
9. $\sqrt{x-3}+\sqrt{x-12}+3=0.$
10. $\sqrt{3x+1}+\sqrt{3x-6}-7=0.$
11. $\sqrt{5x+16}=10-\sqrt{5x-4}.$
12. $\sqrt{4+x}=2\sqrt{1+x}-\sqrt{x}.$
13. $(12+x)^{\frac{1}{2}}+x^{\frac{1}{2}}=2(3+x)^{\frac{1}{2}}.$
14. $\sqrt{1+x}=\dfrac{\sqrt{5+x^2}}{\sqrt{1+x}}.$
15. $\dfrac{3}{\sqrt{x}}+\dfrac{\sqrt{x}}{2}=\dfrac{4}{\sqrt{x}}.$
16. $\dfrac{\sqrt{x}-3}{\sqrt{x}+7}=\dfrac{\sqrt{x}-4}{\sqrt{x}+1}.$
17. $\dfrac{\sqrt{x}+6}{\sqrt{x}+5}=\dfrac{\sqrt{x}+12}{\sqrt{x}+15}.$
18. $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}.$
19. $\dfrac{\sqrt{ax}+b}{\sqrt{ax}-b}=\dfrac{3\sqrt{ax}+5b}{3\sqrt{ax}-2b}.$
20. $\dfrac{\sqrt{3x}-\sqrt{3}}{\sqrt{2x}-\sqrt{2}}=\dfrac{\sqrt{x}+3}{\sqrt{x}+2}.$
21. $\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}}=9.$
22. $\dfrac{2}{x+\sqrt{2-x^2}}+\dfrac{2}{x-\sqrt{2-x^2}}=x.$

## XIII. QUADRATIC EQUATIONS

(Review and Extensions)

**271.** The **degree** of an equation is the greatest exponent or sum of exponents of the unknown numbers in any one term.

**272.** A **quadratic equation** is any equation of the second degree.

Thus, $x^2 + 8x = -12$, $x^2 - 16 = 0$, $x^2 + y^2 = 10$, $xy = 20$ are quadratic equations.

Solve the following by factoring:

1. $x^2 - 7x + 6 = 0$.
2. $x^2 + 8x - 20 = 0$.
3. $x^2 + 9x + 18 = 0$.
4. $x^2 - 5x - 24 = 0$.
5. $4x^2 - 4x - 3 = 0$.
6. $6x^2 - 35x - 6 = 0$.
7. $2x^2 + 13x + 15 = 0$.
8. $6x^2 + 37x + 56 = 0$.

9. Solve $2x^2 - 11x = -16$ by completing the square.

Divide by 2, $x^2 - \frac{11}{2}x = -8$.

Complete square, $x^2 - \frac{11}{2}x + \frac{121}{16} = -8 + \frac{121}{16} = -\frac{7}{16}$.

Extract square root, $x - \frac{11}{4} = \pm\frac{1}{4}\sqrt{-7}$.

Whence, $x = \frac{11}{4} \pm \frac{1}{4}\sqrt{-7}$.

Solve the following by completing the square:

10. $x^2 + 6x = 16$.
11. $x^2 - x = 90$.
12. $x^2 - 8x = 20$.
13. $x^2 + 5x = 14$.
14. $x^2 + 13x = -42$.
15. $x^2 - 19x = -78$.
16. $3x^2 - 2x = 133$.
17. $2x^2 - 5x = 12$.
18. $6x^2 - 37x = -57$.
19. $7x^2 - 5x = 522$.

### EXTRANEOUS ROOTS

**273. Extraneous roots** are the roots that are introduced by clearing an equation of fractions.

The principle of equality conditions, page 143, serves to explain how extraneous roots arise, and to tell which of the roots are extraneous.

**274.** An **integral equation** in $x$ is an equation in which $x$ does not appear in the denominator of any term.

Thus, $\frac{3x}{5} - 7 = \frac{4x}{9}$ is an integral equation in $x$.

**275.** A **fractional equation** in $x$ is an equation in which $x$ does appear in the denominator of one or more of the terms.

Thus, $\frac{5}{x} - \frac{x-2}{x+4} = 8$ is a fractional equation in $x$.

**276.** PRINCIPLES. Let $F = 0$ be any fractional equation in $x$, and suppose that $F = 0$, when multiplied by $M$, becomes the integral equation $I = 0$. Then $I = MF$, and $F = I \div M$. Therefore, § 173,

($a$) If $I = 0$ and $M = 0$ have no root in common, then none of the roots of $I = 0$ are eliminated in passing from $I = 0$ to $F = 0$, and hence no extraneous roots are introduced in passing from $F = 0$ to $I = 0$. That is, the roots of $I = 0$ are the same as those of $F = 0$.

($b$) If $I = 0$ and $M = 0$ have one or more roots in common, say $x = a$ and $x = b$, then these roots are eliminated in passing from $I = 0$ to $F = 0$, and are therefore not roots of $F = 0$; and, conversely, the extraneous roots $a$ and $b$ are introduced in passing from $F = 0$ to $I = 0$. That is, $a$ and $b$ are roots of $I = 0$ but not of $F = 0$.

Therefore, denoting the H. C. F. of $I$ and $M$ by $D$, the extraneous roots introduced in clearing $F = 0$ of fractions are the roots of $D = 0$, and the remaining roots of $I = 0$ are the roots of $F = 0$.

1. Solve $\frac{6}{x} + \frac{2}{x-3} = 1.$ (1)

Multiplying by $M = x(x-3)$, and reducing, we have

$$I = x^2 - 11x + 18 = (x-2)(x-9) = 0. \quad (2)$$

In this case $M$ and $I$ have no common factor; therefore, no extraneous roots are introduced, and the roots of (2), 2 and 9, are the roots of (1).

2. Solve $\frac{4x}{x^2-1} = 1 + \frac{x+1}{x-1}.$ (1)

Multiplying by $M = x^2 - 1$, and reducing, we have

$$I = x^2 - x = x(x-1) = 0. \quad (2)$$

In this case $M$ and $I$ have the factor $x - 1$ in common; therefore one extraneous root, 1, is introduced, and the remaining root of (2), viz. 0, is the only root of (1).

If we attempt to verify (1) for $x = 1$, we obtain $\frac{4}{0} = 1 + \frac{2}{0}$, which has no meaning.

3. Solve $\frac{x-2}{x-5} - \frac{3}{x-2} = \frac{9}{(x-2)(x-5)}.$ (1)

Multiplying by $M = (x-2)(x-5)$, and reducing, we have

$$I = x^2 - 7x + 10 = (x-2)(x-5) = 0. \quad (2)$$

In this case $M$ and $I$ are equivalent, and $D = (x-2)(x-5)$. Therefore two extraneous roots, 2 and 5, are introduced, and as these are the only roots of (2), equation (1) has no root.

Strictly speaking, (1) should not be expressed as an equation; for dividing (2) by $(x-2)(x-5)$ eliminates both equality conditions, and thereby cancels the equality.

Solve, omitting the extraneous roots if any:

1. $\frac{x-2}{x^2-4}=1.$

2. $4x-\frac{14-x}{x+1}=14.$

3. $\frac{1}{(x-1)(x-2)}=\frac{1}{x-1}.$

4. $\frac{x}{5-x}-\frac{5-x}{x}=\frac{15}{4}.$

5. $\frac{3x}{x+5}+\frac{42}{(x+5)(x-2)}=\frac{8x-10}{x-2}.$

6. $\frac{40}{x-5}+\frac{27}{x}=13.$

7. $\frac{(6-x)^2}{6-x}-3=\frac{(x-3)^2}{x-3}.$

8. $\frac{x-\sqrt{x}}{x+\sqrt{x}}=\frac{2}{3}.$

9. $\frac{2x}{x-4}+\frac{2x-5}{x-3}=8\frac{1}{3}.$

10. $\frac{x-\sqrt{x+1}}{x+\sqrt{x+1}}=\frac{5}{11}.$

11. $\frac{x}{x-7}-\frac{7}{x-5}=\frac{14}{(x-7)(x-5)}.$

12. $\frac{2x-3}{3x-5}+\frac{3x-5}{2x-3}=\frac{5}{2}.$

13. $\frac{3x-2}{2x-5}-\frac{2x-5}{3x-2}=\frac{8}{3}.$

14. $\frac{x+3}{x+2}-\frac{2x-3}{x-1}+\frac{x-3}{x-2}=0.$

15. $\frac{a}{x-a}=\frac{2c}{x-c}-\frac{b}{x-b}.$

16. $x^2-\frac{2(a-b)}{a+b}x=\frac{4ab}{(a+b)^2}.$

17. $x^2-\frac{2p\sqrt{q}}{p-q}x=-\frac{pq}{p-q}.$

18. $a^2+b^2-2bx+x^2=\frac{m^2x^2}{n^2}.$

**277. Solving literal quadratic equations.**

1. Solve $x^2 - ax + bx = ab.$

$$x^2 - ax + bx - ab = (x^2 - ax) + (bx - ab)$$
$$= x(x - a) + b(x - a)$$
$$= (x - a)(x + b) = 0.$$

$\therefore$ $x - a = 0;\ x + b = 0;$ whence, $x = a$ or $-b.$

2. Solve $(c + d)^2x^2 - (c^2 - d^2)x = cd.$

Divide by $(c + d)^2$, $x^2 - \frac{c - d}{c + d}x = \frac{cd}{(c + d)^2}.$

Complete square,

$$x^2 - \frac{c - d}{c + d}x + \left(\frac{c - d}{2(c + d)}\right)^2 = \frac{cd}{(c + d)^2} + \left(\frac{c - d}{2(c + d)}\right)^2 = \frac{1}{4}.$$

Extract square root, $x - \frac{c - d}{2(c + d)} = \pm\frac{1}{2}.$

Whence, $x = \frac{c}{c + d}$ or $-\frac{d}{c + d}.$

Solve:

1. $x^2 - 4bx = 12b^2.$
2. $12a^2x^2 - 5ax = 3.$
3. $x^2 - 5nx = 14n^2.$
4. $x^2 - (a + 1)x = -a.$
5. $5ax - 2x^2 = 2a^2.$
6. $x^2 + 2(a + 7)x = -28a.$
7. $x^2 + \frac{b^2 - c^2}{bc}x = 1.$
8. $\frac{a^2x^2}{b^2} + \frac{b^2}{c^2} = \frac{2ax}{c}.$
9. $4x^2 - 6(a + c)x + 2a^2 + 5ac + 2c^2 = 0.$
10. $x^2 - (b - a)c = ax - bx + cx.$

**278. Solving quadratics having irrational roots.**

Find the roots of the following to the nearest hundredth:

1. $x^2 + 2x - 1 = 0.$
2. $x^2 - 3x - 5 = 0.$
3. $x^2 - 2x - 2 = 0.$
4. $x^2 - x - 11 = 0.$
5. $x^2 - 4x + 1 = 0.$
6. $3x^2 - 8x + 2 = 0.$

**279. Solving quadratics by a formula.**

1. Solve $ax^2 + bx + c = 0$.

Transpose $c$, $\quad ax^2 + bx = -c.$

Divide by $a$, $\quad x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Complete square, $\quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$

Reduce, $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$

Extract square root, $\quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$

Whence, $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

By this formula, the values of $x$ in any equation of the form $ax^2 + bx + c = 0$ may be written at once.

2. Solve $2x^2 - 3x - 14 = 0$.

Here $a = 2$, $b = -3$, $c = -14$. Putting these values for the letters in the above formula, we have

$$x = \frac{3 \pm \sqrt{9 + 112}}{4} = \frac{3 \pm 11}{4} = +\frac{7}{2} \text{ and } -2.$$

Solve by the above formula:

1. $3x^2 + 5x - 42 = 0.$
2. $3x^2 + 25x = -8.$
3. $5x^2 + 8x - 21 = 0.$
4. $5x^2 - 46x = -9.$
5. $7x^2 + 4x - 51 = 0.$
6. $10x^2 - 23x = -12.$
7. $2x^2 - 3x - 65 = 0.$
8. $4x^2 + 3x = 115.$
9. $3x^2 - 2x - 85 = 0.$
10. $12x^2 + 481 = 187x.$
11. $\frac{2x^2}{3} - \frac{5x}{2} - \frac{2}{3} = 0.$
12. $\frac{x}{4} - \frac{44}{x - 2} = 4.$
13. $20x^{-2} = 3 - 7x^{-1}.$
14. $2x - 7 + 3x^{-1} = 0.$
15. $3x + 2\sqrt{x} = 16.$
16. $\sqrt{x} + \sqrt{10 - x} = 3\sqrt{2}.$
17. $x^2 + 2rx = n^2 + 2nr.$
18. $x^2 - 4rx = 9n^2 - 12nr.$

**280. Finding the character of the roots of a quadratic without solving the equation.**

**281.** The **discriminant** of a quadratic is that which shows whether the roots are real or imaginary, equal or unequal.

On referring to the formula on page 232 it will be seen that the roots of $ax^2 + bx + c = 0$ will be

($a$) real and unequal, if $b^2 - 4ac$ is positive;

($b$) real and equal, if $b^2 - 4ac$ is zero;

($c$) imaginary, if $b^2 - 4ac$ is negative;

($d$) rational or irrational according as $b^2 - 4ac$ is or is not positive and a perfect square.

Thus, the discriminant of $5x^2 - 7x + 2 = 0$ is $(-7)^2 - 4 \times 2 \times 5$, or $+9$. Therefore the roots are real, rational, and unequal.

The discriminant of $4x^2 - 12x + 9 = 0$ is $(-12)^2 - 4 \times 4 \times 9$, or 0. Therefore the roots are real, rational, and equal.

In like manner find the character of the roots of the following, and check by solving the equation:

1. $x^2 - 7x - 44 = 0$.
2. $5x^2 - x - 42 = 0$.
3. $x^2 + 3x - 70 = 0$.
4. $6x^2 - 7x - 10 = 0$.
5. $x^2 - 8x + 16 = 0$.
6. $7x^2 + 5x - 38 = 0$.
7. $x^2 + 5x + 10 = 0$.
8. $2x^2 - 6x + 5 = 0$.
9. $x^2 - 6x - 13 = 0$.
10. $7x^2 - 8x + 2 = 0$.
11. $2x^2 - 7x - 34 = 0$.
12. $5x^2 - 6x - 41 = 0$.

In the following find the least positive value of $n$ that will render the roots rational:

13. $x^2 - 2nx - 15 = 0$.
14. $x^2 - 8nx - 65 = 0$.
15. $nx^2 + 8x - 3 = 0$.
16. $x^2 + 12x - 4n = 0$.

**282. Forming equations whose roots are given.**

1. Find the equation whose roots are 5 and 8.

Since $x = 5$, $x - 5 = 0$, (1)
and since $x = 8$, $x - 8 = 0$. (2)
(1) × (2), $x^2 - 13x + 40 = 0$, *Ans.*

2. Find the equation whose roots are $-10$ and 6.

Since $x = -10$, $x + 10 = 0$, (1)
and since $x = 6$, $x - 6 = 0$. (2)
(1) × (2), $x^2 + 4x - 60 = 0$, *Ans.*

3. Find the equation whose roots are $r$ and $r'$.

Since $x = r$, $x - r = 0$, (1)
and since $x = r'$, $x - r' = 0$. (2)
(1) × (2), $x^2 - (r + r')x + rr' = 0$. *Ans.*

*Subtract each root from x and place the product of the two remainders equal to* 0; *or*

*For the first term of the equation take $x^2$; for the second term, take x with a coefficient equal to minus the sum of the two roots; and for the third term take the product of the two roots.*

Find the equation whose roots are:

1. 5 and 9.
2. $a + b$ and $+a - b$.
3. 12 and $-7$.
4. $a + b$ and $-a + b$.
5. $-10$ and $+3$.
6. $3c + 1$ and $3c - 1$.
7. $-5\frac{1}{2}$ and $-6\frac{1}{2}$.
8. $3 + \sqrt{2}$ and $3 - \sqrt{2}$.
9. $+\frac{a}{b}$ and $-\frac{b}{a}$.
10. $\frac{a+b}{2}$ and $\frac{a-b}{2}$.
11. $+\frac{3}{4}a$ and $-\frac{2}{3}a$.
12. $\frac{a+b\sqrt{-1}}{2}$ and $\frac{a-b\sqrt{-1}}{2}$.
13. $\frac{a+\sqrt{4b+a^2}}{2}$ and $\frac{a-\sqrt{4b+a^2}}{2}$.

### HIGHER EQUATIONS IN QUADRATIC FORM

**283.** An equation is in the **quadratic form** if it contains only two powers of the unknown number, one of the powers being the square of the other.

Equations in the quadratic form may be solved by completing the square.

1. Solve $x^4 - 10x^2 = -9$.

Since $x^4$ is the square of $x^2$, we may complete the square and proceed as in Art. 149. Thus:

Complete the square, $x^4 - 10x^2 + 25 = -9 + 25 = 16$.
Extract square root, $x^2 - 5 = \pm 4$.
Transpose, etc., $x^2 = 9$ or $1$.
Extract square root, $x = \pm 3$ or $\pm 1$, Ans.

2. Solve $x^{\frac{6}{5}} - 9x^{\frac{3}{5}} = -8$.

Since $x^{\frac{6}{5}}$ is the square of $x^{\frac{3}{5}}$, this equation may be solved in a similar manner.

Complete the square, $x^{\frac{6}{5}} - 9x^{\frac{3}{5}} + (\frac{9}{2})^2 = -8 + \frac{81}{4} = \frac{49}{4}$.
Extract square root, $x^{\frac{3}{5}} - \frac{9}{2} = \pm \frac{7}{2}$.
Transpose, etc., $x^{\frac{3}{5}} = 8$ or $1$.
Extract cube root, $x^{\frac{1}{5}} = 2$ or $1$.
Raise to 5th power, $x = 32$ or $1$, Ans.

NOTE. This equation has six roots; only the two *real* roots are here given.

3. Solve $x + 3\sqrt{x} = 40$.

Write the radical with a fractional exponent,

$$x + 3x^{\frac{1}{2}} = 40.$$

Complete the square, $x + 3x^{\frac{1}{2}} + (\frac{3}{2})^2 = 41\frac{9}{4}$.
Extract square root, $\sqrt{x} + \frac{3}{2} = \pm \frac{13}{2}$.
Transpose, etc., $\sqrt{x} = 5$ or $-8$.
Square, $x = 25$ or $64$.

4. Solve $(x^2-5x)^2-8(x^2-5x)=84$.

Since $(x^2-5x)^2$ is the square of $x^2-5x$, we may proceed as before.

Complete the square, $(x^2-5x)^2-8(x^2-5x)+16=100$.

Extract the square root, $x^2-5x-4=\pm 10$.

Transpose, etc., $x^2-5x=14$, or $-6$.

That is, $x^2-5x=14$, and $x^2-5x=-6$.

Whence, $x=7$ or $-2$, and $x=2$ or $3$.

5. Solve $2x^2+3\sqrt{2x^2+7}=33$.

NOTE. In solving equations of this form, add such a number to both sides that the expression without the radical in the first member may be the same as that within, or some multiple of it.

Add 7 to both sides, $(2x^2+7)+3\sqrt{2x^2+7}=40$.

Complete the square, $(2x^2+7)+3\sqrt{2x^2+7}+(\frac{3}{2})^2=\frac{169}{4}$.

Extract the square root, $\sqrt{2x^2+7}+\frac{3}{2}=\pm\frac{13}{2}$.

Transpose, etc., $\sqrt{2x^2+7}=5$, or $-8$.

Square, $2x^2+7=25$, and $2x^2+7=64$.

Whence, $x=\pm 3$; and $x=\pm\sqrt{\frac{57}{2}}$.

**284.** Only certain forms of equations of the fourth degree can be solved as quadratics; among these are equations of the following form:

$$x^4+2ax^3+(a^2+2b)x^2+2abx=c.$$

If to the first side of this and similar equations we add $b^2$, *or the square of the quotient obtained by dividing the coefficient of* $x$ *by the coefficient of* $x^3$, it will be a perfect square.

Thus, $x^4-16x^3+91x^2-216x=-180$, is of the above form, in which $a=-8$ and $b=13\frac{1}{2}$. In this example the square is completed by adding $(13\frac{1}{2})^2$ to both members.

Solve $x^4 + 10\,x^3 + 17\,x^2 - 40\,x = 84$.

Divide the coefficient of $x$, $-40$, by that of $x^3$, $+10$, and we obtain $-4$. Add the square of $-4$ to both members, and we have

$$x^4 + 10\,x^3 + 17\,x^2 - 40\,x + 16 = 84 + 16 = 100.$$

Extract the square root of both sides,

$$x^2 + 5\,x - 4 = \pm 10.$$

Transpose, etc., $x^2 + 5\,x = 14$, and $x^2 + 5\,x = -6$.

Whence, $x = -7$ or $+2$, and $x = -2$ or $-3$, *Ans.*

Solve:

1. $x^4 - 14\,x^2 = -40$.
2. $x^{10} + 31\,x^5 = 32$.
3. $x^3 + 14\,x^{\frac{3}{2}} = 1107$.
4. $5\,x^{\frac{1}{2}} + x^{\frac{1}{4}} = 22$.
5. $3\,x^{\frac{5}{2}} - 4\,x^{\frac{3}{4}} = 7$.
6. $x^{-1} + x^{-\frac{1}{2}} = 6$.
7. $2\,x + 2 = 5\sqrt{x}$.
8. $4\sqrt[3]{x^2} + 10 = 13\sqrt[3]{x}$.
9. $x^2 - 2\,x + 3 = 2\sqrt{x^2 - 2\,x + 2}$.
10. $x^2 + 5\,x + 4 = 5\sqrt{x^2 + 5\,x + 28}$.
11. $3\,x^2 + 15\,x + 2\sqrt{x^2 + 5\,x + 1} = 2$.
12. $(x + 5)(x - 2) = -3\sqrt{x^2 + 3\,x}$.
13. $(x^2 + 2\,x)^2 - 3(x^2 + 2\,x) + 2 = 0$.
14. $(x^2 - 5\,x)^2 - 8(x^2 - 5\,x) = 84$.
15. $(3\,x^2 - 10\,x + 5)^2 - 8(3\,x^2 - 10\,x + 5) = -12$.
16. $x^4 - 12\,x^3 + 34\,x^2 + 12\,x = 35$.
17. $x^4 - 10\,x^3 + 35\,x^2 - 50\,x = -24$.
18. $x^4 + 6\,x^3 + 5\,x^2 - 12\,x = 12$.
19. $x^4 + 3\,x^3 + 2\,x^2 - \frac{3}{8}\,x = \frac{5}{4}$.

### HIGHER EQUATIONS

**285.** The solution of an equation of a degree higher than the second may often be reduced to the solution of a quadratic by first applying the factor theorem. See § 194.

Solve $x^3 - 5x^2 + x + 15 = 0.$ (1)

By the factor theorem, $x - 3$ is found to be a factor. Dividing the first member of (1) by $x - 3$, gives the quotient $x^2 - 2x - 5$.

$\therefore$ $(x - 3)(x^2 - 2x - 5) = 0.$

Hence, $x - 3 = 0$ and $x^2 - 2x - 5 = 0.$

From $x - 3 = 0$, we obtain $x = 3.$

From $x^2 - 2x - 5 = 0$, we obtain $x = 1 \pm \sqrt{6}.$

Hence, the three roots of the given equation are

$$3,\ 1 + \sqrt{6},\ \text{and}\ 1 - \sqrt{6}.$$

Solve:

1. $x^3 - 6x + 4 = 0.$
2. $x^3 - 7x^2 + 9x - 3 = 0.$
3. $x^3 + 7x + 8 = 0.$
4. $x^3 - 6x^2 + 5x + 12 = 0.$
5. $x^3 - 5x^2 + 6x = 0.$
6. $5x^3 + 2x^2 + 4x + 7 = 0.$
7. $x^3 - 5x + 12 = 0.$
8. $x^3 - 6x^2 + 11x - 6 = 0.$
9. $6x^3 + x^2 - 2x = 0.$
10. $x^4 - 3x^3 + 3x^2 - x = 0.$
11. $6x^3 + 17x^2 + 4x - 12 = 0.$
12. $x^3 - 12x^2 + 20x + 96 = 0.$
13. $x^4 - 5x^3 + 2x^2 + 20x = 24.$
14. $x^4 + 11x^3 + 13x^2 - 85x = 0.$
15. $2x^3 - 13x^2 + 25x - 14 = 0.$
16. $7x^3 - 28x^2 + 35x - 140 = 0.$
17. $5x^4 - 7x^3 + 15x^2 - 21x = 0.$

### SIDES OF RIGHT TRIANGLES

**286.** If the lengths of the hypotenuse, base, and perpendicular of a right triangle are $h$, $b$, and $p$ linear units, respectively, it is shown in geometry that,

$$h^2 = b^2 + p^2.$$

1. The perpendicular of a right triangle is 2 in. longer than the base and the hypotenuse is 10 in. Find the base and perpendicular.

Let $x$ = the number of inches in the base,
then $x + 2$ = the number of inches in the perpendicular.
$\therefore\ x^2 + (x+2)^2 = 10^2.$
Whence, $x = 6$, the base; $x + 2 = 8$, the perpendicular.

2. The base of a right triangle is 5 ft., and the hypotenuse is 1 ft. longer than the perpendicular. Find the perpendicular.

3. The hypotenuse of a right triangle is 8 in. longer than the perpendicular and 4 in. longer than the base. Find the three sides.

4. The perpendicular of a right triangle is 2 in. shorter than the hypotenuse and 7 in. longer than the base. Find the three sides.

5. The perpendicular and base of a right triangle are to each other as 3 to 4, and the hypotenuse is 65 in. Find the two unknown sides.

6. The base and perpendicular of a right triangle are to each other as 3 to 4, and the hypotenuse is 3 ft. longer than the perpendicular. Find the three sides.

7. The perpendicular of a right triangle is 41 in. longer than the base and the hypotenuse is 9 in. longer than the perpendicular. Find the three sides.

### PROBLEMS

1. The result of adding the product of two consecutive integers to their sum is 131. Find the integers.

2. The sum of the squares of two consecutive integers is 211 more than their product. What are the integers?

3. Of four consecutive integers, if the first is multiplied by the fourth and the second by the third, the sum of the products will be 80 more than the sum of the integers. What are the integers?

4. The length of a rectangular field exceeds its breadth by 2 rd., and the area is $1\frac{2}{5}$ A. Find the length of the sides.

5. The length of a rectangular field is twice its breadth. If each of its dimensions were increased by 3 ch., it would then contain 17 A. Find the length of its sides.

6. A man wishes to lay off a rectangular garden such that the diagonal shall be 8 rd. longer than one side and 25 rd. longer than the other. Find the sides of the garden.

7. One train runs 9 mi. per hour faster than another, and covers 216 mi. in 2 hr. less time. Find the speed of each train.

8. A can do a piece of work in 2 hr. less time than B can do it, and together they can do it in $3\frac{3}{7}$ hr. How long will it take each alone to do it?

9. A traveled from C towards D at the rate of 7 mi. a day. After he had gone 32 mi., B set out from D towards C, and went every day $\frac{1}{19}$ of the whole journey; and after he had traveled as many days as he went miles in 1 da., he met A. Find the distance from C to D.

## SIMULTANEOUS QUADRATIC EQUATIONS

**287. A quadratic and a simple equation.**

Solve $\begin{cases} 3x^2 - 4xy + 2y^2 = 6, & (1) \\ 2x + 3y = 7. & (2) \end{cases}$

Solve (2) for $y$, $y = \frac{7 - 2x}{3}$. (3)

Substitute in (1), $3x^2 - 4x\left(\frac{7-2x}{3}\right) + 2\left(\frac{7-2x}{3}\right)^2 = 6$. (4)

Reduce, $59x^2 - 140x + 44 = 0$.

Whence, $x = 2,\ x = \frac{22}{59}$.

Substitute in (3), $y = 1,\ y = \frac{123}{59}$.

*Solve the simple equation for one of the unknown numbers, and substitute in the quadratic.*

Solve:

1. $\begin{cases} x + y = 5, \\ xy = 6. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 25, \\ x + y = 7. \end{cases}$

3. $\begin{cases} x - y = 6, \\ xy = 16. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 325, \\ x - y = 5. \end{cases}$

5. $\begin{cases} x^2 + 2y^2 = 34, \\ x + y = 7. \end{cases}$

6. $\begin{cases} x^2 + xy = 70, \\ x - y = 4. \end{cases}$

7. $\begin{cases} 2x + 3y = 22, \\ 3xy = 60. \end{cases}$

8. $\begin{cases} 7x^2 - 8xy = 159, \\ 5x + 2y = 7. \end{cases}$

9. $\begin{cases} 2x + y = 22, \\ xy + 2y^2 = 120. \end{cases}$

10. $\begin{cases} x^2 + xy + y^2 = 13, \\ 7x + 3y = 24. \end{cases}$

11. $\begin{cases} \frac{3}{x} + \frac{2}{y} = \frac{7}{8}, \\ 5x - 9y = 4. \end{cases}$

12. $\begin{cases} \frac{3x - 2}{y + 5} + \frac{y}{x} = 2, \\ x - y = 4. \end{cases}$

**288. A quadratic and a homogeneous quadratic.**

A **homogeneous quadratic** is one in which all the terms are of the second degree in the unknown numbers.

Thus, $x^2 + 5xy + 7y^2 = 0$, $x^2 - 3xy = 0$, $5xy + 4y^2 = 0$ are homogeneous quadratics.

Solve $\begin{cases} x^2 - 3xy + 2y^2 = 0, & (1) \\ x^2 - 5y = 6. & (2) \end{cases}$

Factor (1), $(x - 2y)(x - y) = 0.$

Whence, $x - 2y = 0,$ (3)

$x - y = 0.$ (4)

Now as in § 287 we solve:

(*a*) (3) and (2), and obtain $x = 4,\ y = 2;\ x = -\frac{1}{4},\ y = -\frac{1}{2}$;

(*b*) (4) and (2), and obtain $x = 6,\ y = 6;\ x = -1,\ y = -1.$

*Solve the homogeneous quadratic for x. Combine each of the simple equations thus obtained with the other quadratic, as in* § 287.

When the homogeneous quadratic cannot be solved by inspection, it may be solved as in § 148 or § 279.

Solve:

1. $\begin{cases} x^2 - 2xy + y^2 = 0, \\ 2x^2 - xy + y = 30. \end{cases}$

2. $\begin{cases} x^2 - xy - 2y^2 = 0, \\ 2x^2 - 3x + y = 3. \end{cases}$

3. $\begin{cases} x^2 - 3xy - 4y^2 = 0, \\ 3x + 5y^2 + 4y = 4. \end{cases}$

4. $\begin{cases} y^2 + 2x^2 = 3xy, \\ y^2 + 2y = x + 6. \end{cases}$

5. $\begin{cases} x^2 - xy = 0, \\ 3x^2 + x - 6y = 2. \end{cases}$

6. $\begin{cases} 3y^2 + 2xy - x^2 = 0, \\ y - 2x + 3x^2 = 32. \end{cases}$

7. $\begin{cases} x^2 - 7xy + 12y^2 = 0, \\ x^2 - 3x + 4y = 4. \end{cases}$

8. $\begin{cases} x^2 - 3xy - 10y^2 = 0, \\ 3x + 2y^2 - 5y = 12. \end{cases}$

**289. Quadratics that are homogeneous only in the unknown numbers.**

Solve $\begin{cases} 2x^2 - xy + 3y^2 = 9, & (1) \\ x^2 - 2xy + 5y^2 = 5. & (2) \end{cases}$

Eliminate the two absolute terms 9 and 5. Thus,

Multiply (1) by 5, $10x^2 - 5xy + 15y^2 = 45.$ (3)

Multiply (2) by 9, $9x^2 - 18xy + 45y^2 = 45.$ (4)

Subtract (4) from (3), $x^2 + 13xy - 30y^2 = 0.$ (5)

Now taking (5) and (1) or (2), and we have the conditions given in § 288.

From (5) we have, $x + 15y = 0,$ (6)

$x - 2y = 0.$ (7)

From (2) and (6), $x = \pm\frac{15\sqrt{13}}{26},\ y = \mp\frac{\sqrt{13}}{26}.$

From (2) and (7), $x = \pm 2,\ y = \pm 1.$

*Combine the given equations so as to eliminate the absolute terms. Combine the homogeneous equation thus obtained with either of the given equations, as in* § 288.

Solve :

1. $\begin{cases} x^2 + xy = 15, \\ xy - y^2 = 2. \end{cases}$

2. $\begin{cases} x^2 + xy + 4y^2 = 6, \\ 3x^2 + 8y^2 = 14. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 20, \\ x^2 - xy = 8. \end{cases}$

4. $\begin{cases} 6x^2 - 5xy + 2y^2 = 12, \\ 3x^2 + 2xy - 3y^2 = -3. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 21, \\ x^2 - xy = 15. \end{cases}$

6. $\begin{cases} x^2 + xy + y^2 = 39, \\ 2x^2 + 3xy + y^2 = 63. \end{cases}$

7. $\begin{cases} xy = 24, \\ x^2 - 2y^2 = 4. \end{cases}$

8. $\begin{cases} x^2 + xy + 2y^2 = 44, \\ 2x^2 - xy + y^2 = 16. \end{cases}$

9. $\begin{cases} 3x^2 + xy = 18, \\ 4y^2 + 3xy = 54. \end{cases}$

10. $\begin{cases} 2y^2 - 4xy + 3x^2 = 17, \\ y^2 - x^2 = 16. \end{cases}$

**290. Solving certain forms of simultaneous equations by special methods.**

The processes consist mainly in reducing the equations to the elementary forms $x+y=a$ and $x-y=b$.

Solve the following:

1. $x^2+y^2=41$, (1) $xy=20$. (2)

(1) + 2 × (2), $x^2+2xy+y^2=81$. (3)

(1) − 2 × (2), $x^2-2xy+y^2=1$. (4)

$\sqrt{(3)}$, $x+y=\pm 9$. (5)

$\sqrt{(4)}$, $x-y=\pm 1$. (6)

From (5) and (6), $x=\pm 5,\ y=\pm 4$.

2. $x+y=7$, (1) $xy=10$. (2)

Square (1), $x^2+2xy+y^2=49$. (3)

(3) − 4 × (2), $x^2-2xy+y^2=9$. (4)

$\sqrt{(4)}$, $x-y=\pm 3$. (5)

From (1) and (5), $x=5$ or 2, $y=2$ or 5.

3. $x^2+y^2=100$, (1) $x+y=14$. (2)

Square (2), $x^2+2xy+y^2=196$. (3)

(3) − (1), $2xy=96$. (4)

(1) − (4), $x^2-2xy+y^2=4$. (5)

$\sqrt{(5)}$, $x-y=\pm 2$. (6)

From (2) and (6), $x=8$ or 6, $y=6$ or 8.

4. $\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{16}$, (1) $\frac{1}{x}+\frac{1}{y}=\frac{3}{4}$. (2)

HINT. Do not clear of fractions, but proceed as in the solution of example 3.

5. Solve $x^3+y^3=152$, (1) $x+y=8$. (2)

Divide (1) by (2), $x^2-xy+y^2=19$. (3)

Square (2), $x^2+2xy+y^2=64$. (4)

Subtract (3) from (4), $3xy=45$. (5)

Divide (5) by 3, $xy=15$. (6)

Combine (2) and (6) as in example 2, $x=5,\ y=3$.

6. Solve $x^4 + z^2y^2 + y^4 = 133$, (1) $x^2 - xy + y^2 = 7$. (2)

Divide (1) by (2), $x^2 + xy + y^2 = 19$. (3)

[(3) + (2)] ÷ 2, $x^2 + y^2 = 13$. (4)

(3) − (2), $2xy = 12$. (5)

Combine (4) and (5) as in example 1, $x = 3, y = 2$.

Solve:

In solving the first two examples proceed as in the solution of example 1, p. 244.

1. $\begin{cases} \frac{1}{x^2} + \frac{1}{y^2} = \frac{29}{100}, \\ \frac{1}{xy} = \frac{1}{10}. \end{cases}$

2. $\begin{cases} x^6 + y^4 = 985, \\ x^3y^2 = 432. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 100, \\ x + y = 14. \end{cases}$

4. $\begin{cases} x^4 + x^2y^2 + y^4 = 931, \\ x^2 + xy + y^2 = 49. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 325, \\ x - y = 5. \end{cases}$

6. $\begin{cases} x^2 + xy + y^2 = 84, \\ x + \sqrt{xy} + y = 14. \end{cases}$

7. $\begin{cases} \frac{1}{x^2} + \frac{1}{y^2} = \frac{13}{36}, \\ \frac{1}{xy} = \frac{1}{6}. \end{cases}$

8. $\begin{cases} \frac{1}{x^2} + \frac{1}{xy} + \frac{1}{y^2} = 49, \\ \frac{1}{x} + \frac{1}{y} = 8. \end{cases}$

9. $\begin{cases} x^4 + y^4 = 17, \\ xy = 2. \end{cases}$

10. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

11. $\begin{cases} x^4 + y^4 = 97, \\ x^2 + y^2 = 13. \end{cases}$

12. $\begin{cases} x + y + \sqrt{x + y} = 12, \\ x^3 + y^3 = 189. \end{cases}$

13. $\begin{cases} x^3 - y^3 = 973, \\ x - y = 7. \end{cases}$

14. $\begin{cases} x + xy + y = 29, \\ x^2 + xy + y^2 = 61. \end{cases}$

15. $\begin{cases} x^5 - y^5 = 31, \\ x - y = 1. \end{cases}$

16. $\begin{cases} \sqrt{x + y} + \sqrt{x - y} = 4, \\ x^2 - y^2 = 9. \end{cases}$

17. $\begin{cases} x^3 - y^3 = 98, \\ x^2y - xy^2 = 30. \end{cases}$

18. $\begin{cases} x^4 + y^4 = 97, \\ 2x^3y - 3x^2y^2 + 2xy^3 = 48. \end{cases}$

### PROBLEMS

1. Find two numbers whose sum is 11 and the sum of whose cubes is 407.

2. Find two numbers whose difference is 2 and the difference of whose cubes is 56.

3. Find two numbers whose sum multiplied by the greater is 84, and whose difference multiplied by the smaller is 10.

4. The product of two numbers is 24, and the difference of their cubes is 19 times the cube of their difference. Find the numbers.

5. A farmer has a rectangular field containing 12 A., and if he makes it 5 ch. wider and 4 ch. longer its area will be doubled. Find the two sides.

6. John rode 12 mi. on his bicycle, when it broke down. He walked back at a speed of 6 mi. less per hour than he rode, and it took him 2 hr. 40 min. longer to walk back than it did to ride out. Find the speeds at which he walked and rode.

7. The sum of the two digits of a number is 14, and the sum of the squares of the digits is 11 more than the number. Find the number.

8. Find the sides of a rectangle whose area is 120 sq. in., and whose diagonal is 17 in.

9. Two men, A and B, traveled from C to D, a distance of 60 mi. They started at the same time, and A reached D 5 hr. in advance of B; but if B's rate per hour had been 2 mi. more and A's rate 1 mi. less, then B would have reached D 2 hr. in advance of A. What were the rates at which A and B traveled per hour?

10. A grocer sold 80 lb. of tea and 100 lb. of coffee for $65; and he sold 60 lb. more of coffee for $20 than he did of tea for $10. What was the price of each?

11. A and B together can do a certain work in 12 da.; if A could do the work in 8 da. less time, and B in 6 da. less time, both together could do it in 8 da. How long will it take each alone to do the work?

12. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yd.; but if the circumference of each wheel is increased one yard, it will make only 4 revolutions more than the hind wheel, in the same distance. What is the circumference of each wheel?

13. A and B are two towns on the bank of a stream, and are 42 mi. apart. A man rows from A to B and back again, and finds, first, that he is 3 hr. 12 min. longer upon the water than he would have been had there been no current; second, that he was 8 hours longer going from A to B than from B to A. Find his rate of rowing, and the rate of the current.

14. A ladder 50 ft. long is leaning against a wall so as to reach a window. Now if the bottom of the ladder is pulled out 16 ft. farther from the wall, it will throw the top of it 8 ft. below the window. What is the height of the window from the ground?

15. Find two numbers whose product equals the difference of their squares, and the sum of whose squares equals the difference of their cubes.

16. Find the price of eggs per dozen when 6 more for 40 cents lowers the price 4 cents per dozen.

The following may be solved by either one or two unknown letters. Let any *even* numbered problem be solved by substituting the given numbers in the answer to the preceding problem.

**17.** Find two numbers whose sum is $s$ and whose product is $p$.

**18.** Find two numbers whose sum is 9 and whose product is $-52$.

**19.** A rectangle is $l$ inches longer than it is wide, and its area is $a$ square inches. Find its sides.

**20.** The difference of the length and width of a rectangle is 7 in., and its area is 120 sq. in. Find its sides.

**21.** Find the sides of a rectangle whose diagonal is $d$ inches, and whose area is $a$ square inches.

**22.** Find the sides of a rectangle whose diagonal is 65 in. and whose area is 1848 sq. in.

**23.** A man paid $d$ dollars for a number of pigs. If the price per pig had been $l$ dollars less, he would have received $p$ pigs more than he did. How many pigs did he buy?

**24.** A lady paid \$6 for some cloth. She would have received 10 yd. more than she did if the price per yard had been 5¢ less. How many yards did she buy?

**25.** By increasing the speed of a train $d$ miles per hour, it requires $a$ hours less time to go $b$ miles. Find the original speed.

**26.** John rode 75 mi. on his bicycle. If his speed per hour had been $2\frac{1}{2}$ mi. more, he would have made the trip in 1 hour less time. Find his speed per hour.

## XIV. RATIO, PROPORTION, AND VARIATION

### RATIO

**291.** A **ratio** is the quotient of one quantity by another quantity of the same kind.

Thus, the ratio of $m$ to $n$ is $\frac{m}{n}$, which is also written $m : n$.

The **terms** of a ratio are the quantities compared; the **antecedent** is the first term or dividend; the **consequent** is the second term or divisor.

Thus, in $\frac{m}{n}$, or $m : n$, $m$ is the antecedent and $n$ the consequent.

**292. Commensurable quantities** are like quantities that contain any third quantity a whole number of times.

Thus, if $A$ and $B$ are two lines such that $A = 5\frac{1}{4}$ in. and $B = 2\frac{1}{3}$ in., $A$ and $B$ are commensurable. For, $A$ contains $\frac{1}{12}$ in. 63 times, and $B$ contains $\frac{1}{12}$ in. 28 times. In this case, $A : B = \frac{63}{28} = \frac{9}{4} = 2\frac{1}{4}$.

*The ratio of two commensurable quantities is a rational number.*

**293. Incommensurable quantities** are like quantities that do not contain any third quantity a whole number of times.

Thus, if the side of a square is $s = 1$ in., then the diagonal of the square is $d = \sqrt{2}$ in. In this case $d : s = \sqrt{2}$.

*The ratio of two incommensurable quantities is an irrational number.*

**294.** *Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.* (§ 111.)

## PROPORTION

**295.** A **proportion** is an equation whose members are ratios.

Thus, $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$

is a proportion, which is read, the ratio of $a$ to $b$ is equal to the ratio of $c$ to $d$.

The same proportion is sometimes written $a : b :: c : d$, which is read, $a$ is to $b$ as $c$ is to $d$.

The **extremes** are the first and last terms, $a$ and $d$, and the **means** are the second and third terms, $b$ and $c$.

**296.** In a proportion like $a : b = c : d$, $d$ is called the **fourth proportional** to $a$, $b$, and $c$.

In a proportion like $a : b = b : c$, $c$ is called the **third proportional** to $a$ and $b$.

In a proportion like $a : b = b : c$, $b$ is called the **mean proportional** between $a$ and $c$.

**297.** A **continued proportion** is a series of equal ratios in which each consequent is the same as the next antecedent.

Thus $a : b = b : c = c : d = d : e$ is a continued proportion.

**298.** The *fundamental test* of a proportion is that the quotient of the first term by the second shall be equal to the quotient of the third by the fourth.

Thus $6 : 8 = 30 : 40$ is a proportion, because $\frac{6}{8} = \frac{30}{40}$, each being equal to $\frac{3}{4}$.

The student should note that the principles of ratios are the same as the properties of equal fractions.

## PRINCIPLES OF PROPORTION

**299. Relation of the means and the extremes.**

(1) *If four numbers are in proportion, then the product of the means is equal to the product of the extremes.*

Let $a : b = c : d$

Then, $\frac{a}{b} = \frac{c}{d}$ Definition.

Multiply by $bd$, $ad = bc$.

ILLUSTRATION. $12 : 3 = 20 : 5$, because $\frac{12}{3} = \frac{20}{5}$; and

$$12 \times 5 = 3 \times 20 = 60.$$

The student should, in like manner, illustrate each of the following principles by one or more numerical examples.

(2) *If the product of two numbers is equal to the product of two other numbers, either pair may be made the means, and the other pair the extremes, of a proportion.*

Let $mq = np$.

Divide by $nq$, $\frac{m}{n} = \frac{p}{q}$; or $m : n = p : q$.

**300. Mean proportional.** *The mean proportional between two numbers is equal to the square root of their product.*

Let $a : b = b : c$.

Then, $b^2 = ac$. $\therefore b = \sqrt{ac}$.

**301. Principle of alternation.** *If four numbers are in proportion, then the first is to the third as the second is to the fourth.*

Let $a : b = c : d$.

Then, $\frac{a}{b} = \frac{c}{d}$.

Multiply by $\frac{b}{c}$, $\frac{a}{c} = \frac{b}{d}$; or $a : c = b : d$.

**302. Principle of inversion.** *If four numbers are in proportion, then the second term is to the first as the fourth term is to the third.*

Let $a : b = c : d.$

Then, $bc = ad.$

Divide by $ac$, $\frac{b}{a} = \frac{d}{c}$; or $b : a = d : c.$

**303. Principle of composition.** *If four numbers are in proportion, then the sum of the first and second terms is to the second as the sum of the third and fourth terms is to the fourth.*

Let $a : b = c : d.$

Then, $\frac{a}{b} = \frac{c}{d}.$

Add 1 to both sides, $\frac{a}{b} + 1 = \frac{c}{d} + 1.$

Reduce, $\frac{a+b}{b} = \frac{c+d}{d}$; or $a + b : b = c + d : d.$

**304. Principle of division.** *If four numbers are in proportion, then the difference of the first and second terms is to the second as the difference of the third and fourth terms is to the fourth.*

HINT. Proceed as in § 303, subtracting 1 from both sides.

**305. Principle of composition and division.** *If four numbers are in proportion, then the sum of the first and second terms is to their difference as the sum of the third and fourth terms is to their difference.*

HINT. By § 303, $\frac{a+b}{b} = \frac{c+d}{d}.$ (1)

By § 304, $\frac{a-b}{b} = \frac{c-d}{d}.$ (2)

Divide (1) by (2), $\frac{a+b}{a-b} = \frac{c+d}{c-d}.$

**306. Products of proportions.** *The products of the corresponding terms of two or more proportions are in proportion.*

Let $$a : b = c : d,$$
$$e : f = g : h,$$
$$k : l = m : n.$$

Then, $$\frac{a}{b} = \frac{c}{d},\ \frac{e}{f} = \frac{g}{h},\ \frac{k}{l} = \frac{m}{n}.$$

Multiplying the first members, and also the second members, of these equations, we have

$$\frac{aek}{bfl} = \frac{cgm}{dhn};\ \text{or}\ aek : bfl = cgm : dhn.$$

**307. Powers and roots.** *Like powers, or like roots, of the terms of a proportion are in proportion.*

Let $a : b = c : d$. Then, $\frac{a}{b} = \frac{c}{d}$.

Raising both sides to the $n$th power, and extracting the $n$th root of both sides, we have, respectively,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n},\ \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}}.$$

$$\therefore\ a^n : b^n = c^n : d^n,\ \sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$$

**308.** *In a* **series of equal ratios,** *the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$$

Place each of these ratios equal to $r$, and reduce,

$$a = br,\ c = dr,\ e = fr,\ g = hr.$$
$$\therefore\ a + c + e + g = (b + d + f + h)r.$$
$$\therefore\ \frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b}.$$
$$\therefore\ a + c + e + g : b + d + f + h = a : b.$$

1. Find $x$ when $6 : 15 = 14 : x$.

HINT. By § 299, $6 \times x = 15 \times 14$.

Find $x$ in the following:

2. $12 : 20 = 15 : x$.
3. $135 : 5x = 72 : 64$.
4. $55 : 25 = x : 35$.
5. $11\frac{1}{4} : 1\frac{5}{6} = 12\frac{6}{7} : x$.
6. $x : 38 = 15 : 19$.
7. $6\frac{3}{7} : 13\frac{1}{3} = x : 11\frac{1}{5}$.
8. $2\frac{1}{4} : x = 26\frac{1}{4} : 6\frac{2}{3}$.
9. $46.2 : 7.7 = 5.4 : x$.

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10. $x + 2 : x - 3 = 3 : 2$.
11. $x + 16 : 9 = x - 12 : 2$.
12. $x + 3 : x - 3 = 5 : 3$.
13. $2x + 3 : 3 = 3x - 1 : 4$.

Find the fourth proportional to:

14. 3, 4, 5.
15. 18, 15, 12.
16. $ab$, $ac$, $bd$.
17. 8, 6, 4.
18. $3\frac{1}{3}$, $4\frac{1}{6}$, $6\frac{1}{2}$.
19. $a^3$, $a^2b$, $ab^2$.

Find the third proportional to:

20. 18 and 12.
21. $a$ and $x$.
22. $8a^2b$ and $5\frac{1}{7}ab^2$.
23. $3\frac{1}{2}$ and $5\frac{5}{6}$.
24. $\frac{2}{3}$ and $\frac{3}{4}$.
25. $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.

Find the mean proportional between:

26. 25 and 16.
27. $a^2$ and $b^4$.
28. $+a^2$ and $-b^2$.
29. $5\frac{4}{9}$ and $2\frac{7}{9}$.
30. $\frac{ab}{c}$ and $\frac{ac}{b}$.
31. $+6\frac{1}{4}$ and $7\frac{1}{9}$.

Solve for $x$ and $y$:

32. $x + 2 : y + 1 = 3 : 4$, and $x + 6 : y + 5 = 5 : 6$.
33. $x + 7 : y + 9 = 3 : 4$, and $x - 4 : y - 5 = 2 : 3$.
34. $4x + 1 : 5y - 2 = 6 : 8$, and $7x - 3 : 9y + 2 = 4 : 7$.
35. $ax + c : by - d = 2 : 3$, and $mx - n : py + q = 3 : 4$.

### PROBLEMS

1. Two numbers are as 3 to 4, and if 7 is subtracted from each, the remainders are as 2 to 3. Find the numbers.

2. What number must be added to each of the numbers 5, 9, 20, 29 in order that the four sums shall be proportional?

3. A father says to his son, "8 yr. ago our ages were as 10 to 3, and 12 yr. from now they will be as 15 to 8." Find their present ages.

4. Find two numbers whose sum is 8, and whose product is to the sum of their squares as 15 to 34.

5. Show how the solution of $x^2+x+1 : x^2-x+1 = 2x+1 : 2x-1$ may be simplified by § 305.

6. Two numbers are in the ratio of 2 to 3, and if 9 is added to each they will be in the ratio of 3 to 4. Find the numbers.

7. There are two numbers whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers?

8. Of two square grassplots, a side of one is 10 yd. longer than the side of the other, and their areas are as 25 to 9. Find the length of the sides.

9. There are two numbers whose sum is to the less as 5 to 2, and whose difference is to 9 as 15 is to the difference of their squares. Find the numbers.

10. There are two numbers whose sum is 10, and the difference of whose fourth powers is to the sum of their squares as 26 is to 7. Find the numbers.

### RECTANGULAR SOLIDS

**309.** If a rectangular solid, like a box, is $a$ units long, $b$ units wide, and $c$ units high, and contains $V$ cubic units, we have

$$V = abc.$$

1. Interpret this formula, calling $V$ the volume.

2. Find the volume of a rectangular solid whose dimensions are 12, 10, and 6 in.

3. The dimensions of a box are proportional to 2, 3, 4, and its volume is 648 cu. in. Find its dimensions. Its entire surface.

4. Deduce a formula for the total surface ($S$) of the solid of which $V = abc$.

5. A box is 3 in. wider than it is high, and 2 in. longer than it is wide, and its total surface is 180 sq. in. Find its three dimensions.

6. A box is 3 in. higher than it is wide and 4 in. longer than it is high, and its volume is 480 cu. in. Find its three dimensions. See § 285.

7. A rectangular sheet of tin, 16 by 20 in., is made into an open box by cutting out a square from each corner and turning up the sides. Find the size of the square cut out if the volume of the box is 420 cu. in.

8. The areas of the bottom, side, and end of a box are, respectively, $B$, $S$, and $E$ sq. in. Find the volume of the box.

9. A box, whose length, width, and depth are proportional to 6, 4, and 3, would contain 5400 cu. in. more than it does if it were 4 in. wider and 5 in. deeper. Find the dimensions of the box.

## VARIATION

**310. Variables and constants.** The length of the *shadow* of a flagstaff *varies* from one value to another, but the length of the *flagstaff* itself undergoes *no change.*

When a ball is thrown upward its *height* from the ground *changes* from one value to another, but the *size* of the ball undergoes *no change.*

**Variables** are quantities that change from one size or value to another, and **constants** are quantities that undergo no change of size or value.

Variables are usually represented by the last letters of the alphabet, $u$, $v$, $w$, $x$, $y$, $z$; and constants by figures or by the first letters of the alphabet, $a$, $b$, $c$, $\cdots$, $k$, $l$.

If a variable is denoted by $y$, particular values of the variable are denoted by $y_1$, $y_2$, etc.

Thus, if $y$ = the number of ounces in the weight of a growing apple, and the apple at one time weighs 5 oz., and at another time 9 oz., we would write $y_1 = 5$, $y_2 = 9$.

*Particular values of variables are constants.*

**311.** If $x$ = the side of a square and $y$ = its area, then $y = x^2$. Now as $x$ changes from $x_1 = 1$ to $x_2 = 2$, $y$ will change from $y_1 = 1$ to $y_2 = 4$. That is, a change in the value of $x$ produces a change in the value of $y$.

When one variable, as $y$, is so related to another variable, $x$, that any change in the value of $x$ produces a change in the value of $y$, $y$ is called a **function** of $x$.

Thus, if $y = 3x + 4$, or $y = x^2 - 5x + 6$, or $y = 3$, $y$ is a function of $x$.

### CORRESPONDING VALUES

**312.** If $y$ is a function of $x$, as $y = x^2 - 3x + 5$, and any particular value is assigned to $x$, as $x_1 = 2$, the result of substituting 2 for $x$ in the given function, viz. 3, is called the corresponding value of $y$; that is, $y_1 = 3$.

Thus, in $y = 6x - 7$, corresponding values of $x$ and $y$ are, respectively, 1 and $-1$, 2 and 5, 3 and 11, ....

### PROPORTIONAL VARIATION

**313.** If two variables are so related that any two values of the one are proportional to the corresponding values of the other, then either variable is said to **vary as** the other.

Thus, if $y$ is a function of $x$ such that

$$y : y_1 = x : x_1,$$

$y$ and $y_1$ being corresponding values of $x$ and $x_1$, then $y$ varies as $x$; and this relation is written

$$y \propto x.$$

**314.** *If $y \propto x$, then $y$ equals $x$ multiplied by some constant.*

PROOF. Since $y \propto x$,

then $y : y_1 = x : x_1$.

Whence, $y = \frac{y_1}{x_1} x$.

Denoting the constant ratio $\frac{y_1}{x_1}$ by $c$, we have

$$y = cx.$$

The expression $y \propto x$ is equivalent to $y = cx$, or $y = mx$, or $y = nx$, where $c$, $m$, $n$ are constants; and, conversely, $y = cx$, or $y = mnx$, etc., may be written $y \propto x$.

**315. Direct variation.** $y \propto x$ expresses, and is read, "$y$ varies as $x$."

ILLUSTRATION. The distance ($s$) covered by a train moving at a constant rate of speed ($r$) varies directly as the time ($t$). That is,

$$s \propto t, \text{ or } s = rt, \text{ or } s : s_1 = t : t_1.$$

**316. Inverse variation.** $y \propto \frac{1}{x}$ expresses, and is read, "$y$ varies inversely as $x$," or "$y$ varies as the reciprocal of $x$."

ILLUSTRATION. The number of men ($x$) required to do a certain work varies inversely as the time ($t$). That is

$$x \propto \frac{1}{t}, \text{ or } x = \frac{c}{t}, \text{ or } x : x_1 = \frac{1}{t} : \frac{1}{t_1}.$$

In this case the constant $c$ is the total number of units of work.

**317. Joint variation.** $u \propto xy$ expresses, and is read, $u$ varies jointly as $x$ and $y$.

ILLUSTRATION. The cost ($u$) of a bolt of cloth varies jointly as the number of yards ($y$) in the bolt and the price per yard ($x$). That is,

$$u \propto xy, \text{ or } u = cxy, \text{ or } u : u_1 = xy : x_1y_1.$$

**318. Direct and inverse variation.** $u \propto \frac{x}{y}$ expresses, and is read, $u$ varies directly as $x$ and inversely as $y$.

ILLUSTRATION. In mowing a field, the time ($t$) required for the work varies directly as the number of acres ($x$), and inversely as the number of men ($y$) engaged in the work. That is,

$$t \propto \frac{x}{y}, \text{ or } t = \frac{mx}{y}, \text{ or } t : t_1 = \frac{x}{y} : \frac{x_1}{y_1}.$$

**319. Compound variation.** $u \propto mx^2 + \frac{n}{x^3}$ expresses, and is read, $u$ varies as the sum of two numbers, one of which ($mx^2$) varies directly as $x^2$ and the other $\left(\frac{n}{x^3}\right)$ inversely as $x^3$.

If $y$ varies as $x$, and if $y = 15$ when $x = 3$, find $y$ when $x = 8$.

Since, $y \propto x$, $\therefore y = mx$.
Now when $x = 3$, $y = 15$. $\therefore 3m = 15$, and $m = 5$.
The relation, then, of $y$ to $x$ is $y = 5x$.
Therefore, when $x = 8$, $y = 40$.

1. If $y$ varies as $x$, and if $y = 40$ when $x = 5$, find $y$ when $x = 12$.

2. If $y \propto x$, and $y = 27$ when $x = 3$, find $x$ when $y = 45$.

3. If $y \propto xz$, and $y = 42$ when $x = 2$, $z = 3$, find $y$ when $x = 4$ and $z = 5$.

4. If $y$ varies inversely as $x$, and if $y = 6$ when $x = 4$, find $x$ when $y = 4\frac{1}{5}$.

5. If $y$ varies directly as $x$ and inversely as $z$, and $x = 5$ when $y = 30$ and $z = 7$, find $z$ when $y = 35$ and $x = 10$.

6. If $z \propto x + y$, and $z = 15$ when $x = 3$ and $y = 2$, find the value of $z$ in terms of $x$ and $y$.

7. If $x^2 \propto y^3$, and $x = 4$ when $y = 2$, find the value of $y$ in terms of $x$.

8. If $x \propto y^2$, and $y \propto z^3$ and $z = 1$ when $y = 2$ and $x = 8$, find the value of $x$ when $z = 2$.

9. If one number varies directly as another, and the former is $\frac{3}{4}$ when the latter is $\frac{4}{5}$, what will the latter be when the former is 9?

10. Given that the area of a circle varies as the square of its radius, prove that the area of a circle whose radius is 5 in. is equal to the sum of the areas of two circles whose radii are 3 and 4 in.

**11.** Given that the volume of a sphere varies as the cube of its radius, prove that the volume of a sphere whose radius is 6 in. is equal to the sum of the volume of three spheres whose radii are 3, 4, and 5 in.

**12.** The distance fallen by a body from rest varies as the square of the time of falling. If a body falls $257\frac{1}{3}$ ft. in 4 sec., how far will it fall in 10 sec.?

**13.** The volume of a right circular cone varies jointly as its height and the square of the radius of its base. If the volume of a certain cone is 100 cu. in., what is the volume of another cone having 3 times the altitude and $\frac{1}{3}$ of the base radius?

**14.** The volume of a mass of gas varies as the absolute temperature and inversely as the pressure. When the absolute temperature in a given case is 240 and the pressure 18, the volume is 160 cu. in. What will be the volume when the absolute temperature becomes 300 and the pressure 20?

**15.** The pressure on the horizontal head of a cylindrical drum immersed in water varies jointly as the depth of the head below the surface and the square of the radius of the head. If the pressure is 1500 lb., when the depth is 15 ft. and the radius of the head 3 ft., find the pressure when the depth is 18 ft. and the radius 5 ft.

**16.** The illumination of a book by a lamp varies inversely as the square of the distance of the book from the light. If the illumination at 30 in. is 32 candle power, what candle power will fall on the book at the distance of 4 ft.?

### PHYSICAL FORMULAS INVOLVING QUADRATICS

**320.** In the following formulas $s$ = distance, $v$ = velocity, $t$ = time, and $g = 32.16$ = acceleration of gravity, or increase of velocity in feet per second.

The formulas for uniformly accelerated motion are:

$$v = gt. \qquad (1) \qquad\qquad s = \tfrac{1}{2} gt^2. \qquad (2)$$

These formulas are slightly inaccurate on account of the resistance of the atmosphere.

1. Give the interpretation of (1); of (2).

2. How does $v$ vary? How does $s$ vary?

3. If a body falls from rest, what will be its velocity at the end of 8 seconds?

4. Through what distance will a stone fall in 5 seconds?

5. Solve (1) for $t$, and give the interpretation.

6. How long, to the nearest tenth of a second, must an iron ball be in falling, to have a velocity of 1000 ft. per second?

7. Solve (2) for $t$, and give the interpretation.

8. How long, to the nearest tenth of a second, will it take a marble to fall 1000 ft.?

9. A boy dropped a stone in a well, and the stone was 3 seconds in falling. Find the depth of the well.

10. Eliminate $t$ between (1) and (2). Solve the result for $s$, and give the interpretation.

11. Through what distance has a body fallen when its velocity is 500 ft. per second?

## XV. SERIES

### THE $n$th TERM OF A SERIES

**321.** In what follows $n$ stands for any positive integer. The successive values of $n$ are 1, 2, 3, ···, and a **series*** is the corresponding successive values of any function of $n$.

Thus, substituting 1, 2, 3, ··· for $n$ in $6n - 3$, we obtain the *series* 3, 9, 15, ···, whose terms are 3, 9, 15, etc.

**322.** The function from which the series is derived is called the **$n$th term**, or **last term** of the series. It is also called the **law of the series** because its interpretation gives the rule by which the terms are formed.

Thus, $6n - 3$ being the function, the interpretation is, any term of the series is 3 less than 6 times the number of the term.

The first and last terms of a series are called the **extremes**, and all the intermediate terms the **means**.

**323. Notation.** The number of terms in a series is denoted by $n$, the first term by $a$, the last term by $l$, and the sum of all the terms by $s$.

The $n$th term is often denoted by $a_n$, in which case $a_1$ is the first term, $a_2$ the second, ···, $a_n$ the last or $l$.

Thus, if $a_n = n(n + 1)$, $a_1 = 2$, $a_2 = 6$, $a_3 = 12$, etc. If the series is continued until there are 10 terms, then $a_{10} = l = (10)(10 + 1) = 110$.

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* It must not be inferred that all series are of this character. This merely defines the kind of series treated in this book.

Find the first ten terms of a series of which $a_n = \frac{1}{2}n(n-1)$, and give the law of formation.

Making $n = 1, 2, 3, \cdots 10$, we have $a_1 = 0$, $a_2 = 1$, $a_3 = 3$, $a_4 = 6$, $a_5 = 10$, $a_6 = 15$, $a_7 = 21$, $a_8 = 28$, $a_9 = 36$, $a_{10} = 45$.

LAW. Any term is found by multiplying $\frac{1}{2}$ of the number of the term by the number less one.

Solve the following in a similar manner, giving the law of formation:

1. $a_n = 5n - 3$.
2. $a_n = n^2 + 3n + 2$.
3. $a_n = 4n - 9$.
4. $a_n = n^2 - 3n + 2$.
5. $a_n = (n-4)n$.
6. $a_n = n^2 + 5n + 6$.
7. $a_n = n^2 - 25$.
8. $a_n = n^2 - 7n + 10$.
9. $a_n = \dfrac{n}{n+1}$.
10. $a_n = \dfrac{n(n+1)(n+2)}{6}$.
11. $a_n = \dfrac{n(n+1)}{2n+3}$.
12. $a_n = \dfrac{n(n+2)(n+4)}{(n+1)(n+3)}$.
13. $a_n = 2^n - 1$.
14. $a_n = 3 \times 2^{n-1} - 5$.

**324.** When several terms of a series are given, the $n$th term or law of the series can often be found by inspection.

Find the $n$th term of 5, 12, 21, 32, ....

Factoring the terms, $1 \times 5,\ 2 \times 6,\ 3 \times 7,\ 4 \times 8$

which may be written, $1 \times (1+4),\ 2 \times (2+4),\ 3 \times (3+4)$.

That is, any term is equal to the number of the term multiplied by the number plus 4. $\therefore a_n = n(n+4)$.

Find the $n$th term of:

1. 3, 8, 15, 24, ....
2. 0, 2, 6, 12, ....
3. 6, 7, 8, 9, ....
4. 5, 8, 11, 14, ....
5. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$.
6. $\frac{6}{3}, \frac{7}{4}, \frac{8}{5}, \frac{9}{6}, \cdots$.
7. $3^2, 4^2, 5^2, \cdots$.
8. $3, 3 \times 2, 3 \times 2^2, \cdots$.

## ARITHMETIC PROGRESSION

**325.** An **arithmetic progression** (A. P.) is a series in which each term is obtained from the preceding term by the addition of a positive or a negative constant, called the **common difference** (denoted by $d$).

Thus, 5, 8, 11, $\cdots$ is an A. P. in which $a = 5$ and $d = 3$;
7, 5, 3, $\cdots$ is an A. P. in which $a = 7$ and $d = -2$.

**326. Finding the $n$th term of an A. P.**

The first term being $a$ and the common difference $d$, the A. P. is

$$a_1 = a,\ a_2 = a + d,\ a_3 = a + 2d,\ a_4 = a + 3d,\ \cdots.$$

Evidently the coefficient of $d$ in any term is 1 less than the number of the term.

$\therefore$ $$a_n = a + (n-1)d, \qquad \text{(A)}$$

or (§ 323) $$l = a + (n-1)d. \qquad \text{(B)}$$

1. Give the interpretation of formula (A).

2. Find the 20th term of 5, 8, 11, $\cdots$.

Here $a = 5$, and $d = 3$. Therefore by (A) $a_{20} = 5 + 19 \times 3 = 62$.

3. Find the 30th term of 4, 9, 14, $\cdots$.

4. Find the 10th term of 23, 21, 19, $\cdots$.

5. Find the 13th term of $3\frac{1}{2}$, 5, $6\frac{1}{2}$, $\cdots$.

6. Find the 15th term of 31, 28, 25, $\cdots$.

7. When $a$ is 11 and $d$ is 5, what is $a_9$?

8. When $a$ is $2\frac{1}{4}$ and $d$ is $\frac{3}{8}$, what is $a_{43}$?

9. In a certain A. P. $a_n = 7 - 3n$. Find $a$ and $d$.

10. In a certain A. P. $a_n = \frac{5}{6}(n+1)$. Find $a$ and $d$.

11. Find the $n$th term of $7\frac{1}{2}$, $5\frac{3}{4}$, 4, $\cdots$.

**12.** Solve (B) for $d$, and give the interpretation.

**13.** Find $d$ when $a = 5$, $n = 21$, and $l = 45$.

**14.** Solve (B) for $a$, and give the interpretation.

**15.** Find $a$ when $d = 4$, $n = 46$, and $l = 187$.

**16.** Solve (B) for $n$, and give the interpretation.

**17.** Find $n$ when $a = 15$, $d = -3$, and $l = -30$.

**18.** Find the 25th term of an A. P. in which the 4th term is 15 and the 9th term is 35.

**19.** Given $a_4 = 15$, $a_9 = 35$. Find $a_{25}$.

By (A), $a_4 = a_1 + 3d$, $a_9 = a_1 + 8d$, $a_{25} = a_1 + 24d$.

$\therefore$ $a + 3d = 15$, $a + 8d = 35$.

Whence, $a = 3$, $d = 4$.

$\therefore$ $a_{25} = a_1 + 24d = 3 + 24 \times 4 = 99$.

**20.** If $a_5 = 3\frac{1}{2}$, $a_{12} = 7$, what is $a_{20}$?

**21.** Given $a_3 = 5$ and $a_{14} = 27$. Find $a_n$, and by it obtain the first 9 terms of the A. P.

**22.** The $n$th term of an A. P. is $p + qn$. Find the first term and the common difference.

**327. Finding arithmetic means.**

Find 4 arithmetic means between 3 and 35.

Including the 4 means and 2 extremes there will be 4 + 2 or 6 terms in all, of which $a_1 = 3$ and $a_6 = 35$. Then $a_1 = 3$ and $a_1 + 5d = 38$. Whence, $d = 7$.

Therefore, the A. P., showing the four means, is

3, [10, 17, 24, 31], 38.

**1.** Insert 5 arithmetic means between 4 and 58.

**2.** Insert 7 arithmetic means between $-5$ and 19.

**3.** Insert 9 arithmetic means between 3 and 4.

**4.** Insert 15 arithmetic means between $4\frac{1}{6}$ and $5\frac{1}{2}$.

**328. Finding the sum of $n$ terms of an A. P.**

By definition and notation (§§ 325, 323) we have

$s = a + (a+d) + (a+2d) \cdots (l-2d) + (l-d) + l \cdots$, (1)

or $s = l + (l-d) + (l-2d) \cdots (a+2d) + (a+d) + a \cdots$. (2)

(2) is (1) written in inverse order.

Adding (1) and (2),

$$2s = (a+l) + (a+l) + \cdots \text{ to } n \text{ terms}$$
$$= n(a+l)$$
$$\therefore \quad s = \frac{n}{2}(a+l). \qquad \text{(C)}$$

1. Give the interpretation of this formula.

2. Find the sum of an A. P. of 12 terms, the first term being 8 and the last 85.

3. Find the sum of 20 terms of 13, 16, 19, ....

HINT. First find $l$ by (B).

4. Find the sum of 31 terms of 9, 17, 25, ....

5. Find the sum of 25 terms of 33, 29, 25, ....

6. Find the sum of 16 terms of $5\frac{1}{3}$, 6, $6\frac{2}{3}$, ....

7. Find the sum of 100 terms of 1, 2, 3, ....

8. Find the sum of $n$ terms of 1, 3, 5, ....

9. Find the sum of 15 terms of $\frac{1}{2}$, $1\frac{1}{4}$, 2, ....

10. The addends in the following form an A. P. Why? Add them by (C), and check by ordinary addition.

| 11. | 12. | 13. | 14. |
|---|---|---|---|
| 11¢ | $146x$ | $395m$ | \$8.45 |
| 18¢ | $271x$ | $858m$ | \$9.81 |
| 25¢ | $396x$ | $1321m$ | \$11.17 |
| 32¢ | $521x$ | $1784m$ | \$12.53 |
| 39¢ | $646x$ | $2247m$ | \$13.89 |

15. If a body falls $16\frac{1}{12}$ ft. the first second, and in each succeeding second $32\frac{1}{6}$ ft. more than in the next preceding one, how far will it fall in $n$ seconds? in 10 sec.?

**329.** In the two equations

$$l = a + (n-1)d \qquad (B) \qquad s = \frac{n}{2}(a+l) \qquad (C)$$

there are five letters, $a$, $d$, $l$, $n$, $s$. If any three of these are given, the equations may be solved for the other two.

1. How many terms of the A. P. 7, 9, 11, $\cdots$ must be taken in order that their sum shall be 216?

Here $a = 7$, $d = 2$, $s = 216$.

Substituting in (B) and (C),

$$l = 5 + 2n, \qquad (1) \qquad 216 = \frac{n}{2}(7 + l). \qquad (2)$$

Substituting the value of $l$ in (1) for $l$ in (2), and reducing, we have

$$n^2 + 6n = 216. \qquad \therefore n = -18, \text{ and } n = 12.$$

Since $n$ is to be always a positive integer (§ 321), $n = 12$ is the only solution.

2. Given $n = 10$, $l = 29$, $s = 155$. Find $d$.

For these values (B) is $29 = a + 9d$ (1)

and (C) is $155 = 5(a + 29)$. (2)

From (2) we find $a = 2$. Then from (1) we find $d = 3$.

1. The last term of an A. P. is 50, the common difference 4, and the sum of the terms 338. How many terms are there?

2. Given $a = 3$, $d = 6$, $s = 507$. Find $n$.

3. Given $a = 3$, $l = 45$, $d = 2$. Find $s$.

4. Given $a = 1$, $l = 50$, $s = 204$. Find $d$.

5. Given $d = 4$, $l = 35$, $s = 161$. Find $a$ and $n$.

6. The first term of an A. P. is $3\frac{1}{2}$ and $d$ is $\frac{3}{4}$. What is the number of the term that is 17?

**330.** In solving problems in arithmetic progression it is often best to represent the series by:

($a$) $x-y,\ x,\ x+y$, when there are three terms;

($b$) $x-2y,\ x-y,\ x,\ x+y,\ x+2y$, when there are five terms;

($c$) $x-3y,\ x-y,\ x+y,\ x+3y$, when there are four terms. Note that in ($a$) and ($b$) the common difference is $y$, and in ($c$) it is $2y$.

There are three numbers in A. P., whose sum is 15 and the sum of whose squares is 83. Find the numbers.

Let $x-y,\ x,\ x+y$ = the three numbers.

Then $x-y+x+x+y=15$, or $3x=15$, (1)

and $(x-y)^2+x^2+(x+y)^2=83$, or $3x^2+2y^2=83$. (2)

From (1), $x=5$; then from (2), $y=2$.

Therefore the numbers are $5-2,\ 5,\ 5+2$, or 3, 5, 7.

1. How many integers between 70 and 700 are exactly divisible by 6?

HINT. $a=72,\ l=696,\ d=6$.

2. What must be the value of $x$ so that $3x-2$, $5x+11$, and $12x+9$ shall be in A. P.?

HINT. $(5x+11)-(3x-2)=(12x+7)-(5x+11)$. Why?

3. The sum of the 5th and 9th terms of an A. P. is 38, and the sum of the 7th and 15th terms is 62. Find the $n$th term, and by it find the first four terms.

HINT. $a_5+a_9=38$; that is, $a_1+4d+a_1+8d=38$;

$a_7+a_{15}=62$; that is, $a_1+6d+a_1+14d=62$. (A)

4. The sum of the first two terms of an A. P. is 4, and the 5th term is 9. Find the series.

5. The last term of an arithmetic progression is 52, the common difference 5, and the sum of the series 297; find the first term and the number of terms.

6. There are 4 numbers in arithmetic progression; their sum is 32, and their product 3465. Find the numbers.

7. The sum of 3 numbers in arithmetic progression is 30, and the sum of their squares 308. Find the numbers.

8. The sum of the 4th powers of three consecutive integers is 353. Find the integers.

9. The product of 4 consecutive integers is 1680. Find the integers.

10. The sum of 7 numbers in arithmetic progression is 35, and the sum of their cubes is 1295. Find the numbers.

11. Two travelers (A and B) set out from the same place at the same time; A travels uniformly at the rate of 3 miles an hour, but B's rate is 4 miles the first hour, $3\frac{1}{2}$ the second, 3 the third, and so on. In how many hours will A overtake B?

12. The sum of 6 terms of an arithmetic progression is 93, and the 6th term is 28. What is the series?

13. A traveler, A, sets out for a certain place, and goes 1 mile the first day, 2 miles the second day, 3 the third, and so on. In 5 days afterward B sets out, and travels 12 miles a day. How long and how far must B travel before he and A will be together?

14. Two hundred stones being placed on the ground in a straight line, at the distance of 2 feet from each other, how far will a person travel who shall bring them separately to a basket, which is placed 20 yards from the first stone, if he starts from the spot where the basket stands?

## GEOMETRIC PROGRESSION

**331.** A **geometric progression** (G.P.) is a series in which each term is obtained from the preceding term by multiplying it by a positive or negative constant, called the **ratio** (denoted by $r$).

Thus, 2, 6, 18, ··· is a G. P. in which $a = 2$ and $r = 3$.
8, $-4$, 2, ··· is a G. P. in which $a = 8$ and $r = -\frac{1}{2}$.

**332. Finding the $n$th term of a G.P.**

The first term being $a$ and the ratio $r$, the G. P. is

$$a_1 = a,\ a_2 = ar,\ a_3 = ar^2,\ a_4 = ar^3, \text{ etc.}$$

Evidently the exponent of $r$ in any term is 1 less than the number of the term.

$$\therefore a_n = ar^{n-1} \qquad \text{(A)}$$

$$\text{or} \qquad l = ar^{n-1} \qquad \text{(B)}$$

1. Give the interpretation of formula (A).
2. Find the 7th term of 5, 15, 45, ···.

Here $a = 5$ and $r = 3$. Hence, by (B), $a_7 = 5 \times 3^6 = 3645$.

3. Find the 9th term of 5, 20, 80, ···.
4. Find the 6th term of 48, 24, 12, ···.
5. Find the 10th term of 5, 10, 20, ···.
6. When $a$ is 135 and $r$ is $\frac{1}{3}$, what is $a_5$?
7. When $a = 28{,}672$ and $r = \frac{1}{4}$, $a_8 = ?$
8. Find the 6th term of the G. P. $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$.
9. Find the 9th term of the G. P. 1, $-\frac{1}{2}$, $+\frac{1}{4}$, ···.
10. Find the 8th term of the G. P. $-21$, 14, $-9\frac{1}{3}$, ···.
11. In a certain G. P. $a_n = 5 \times 2^{n+3}$. Find $a$ and $r$.
12. Find the $n$th term of 3, 15, 75, 375, ···.

Solve (B) for $r$, and give the interpretation.

Divide (B) by $a$, . $\quad r^{n-1} = \frac{l}{a}$.

Extract the $(n-1)$th root, $\quad r = \sqrt[n-1]{\frac{l}{a}}$.

INTERPRETATION. *The ratio is the result of dividing the last term by the first and taking that root of the quotient whose index is the number of terms less one.*

1. In a G. P. $a = 7$, $l = 112$, and $n = 5$. Find $r$.

2. In a G. P. $a = \frac{2}{3}$, $l = \frac{6}{2}\frac{4}{5}$, and $n = 3$. Find $r$.

**333.** When $a$, $l$, and $r$ are given, the only way, at present, to find $n$ is by trial.

Thus, if $a = 5$, $l = 32{,}805$, and $r = 3$, then by $(B)$,

$$5 \times 3^{n-1} = 32{,}805.$$

Divide by 5, $\quad 3^{n-1} = 6561.$

Now raising 3 to the 2d, 3d, 4th, etc., powers, we find $3^8 = 6561$.
$\therefore n - 1 = 8$, and $n = 9$.

1. In a G. P. $a = 11$, $l = 2816$, and $r = 2$. Find $n$.

2. Find the 10th term of a G. P. in which the 4th term is 72 and the 9th term is 2304.

By (A), $\quad a_4 = ar^3$, $a_9 = ar^8$, $a_{10} = ar^9$.

$\therefore$ $\quad ar^3 = 72$, (1) $\quad ar^8 = 2304$. (2)

Dividing (2) by (1), $\quad r^5 = 32$. $\therefore r = 2$, and $a = 9$.

Then $\quad a_{10} = ar^9 = 9 \times 2^9 = 4608$.

3. In a G. P. $a_3 = 375$ and $a_6 = -46{,}875$. Find $a_8$.

4. In a G. P. $a_5 = \frac{1}{10}$ and $a_9 = \frac{1}{1600}$. Find $a_n$, and by it obtain the first 5 terms of the G. P.

5. The $n$th term of a G. P. is $5 \times 3^{2n-3}$. Find the first term and the ratio.

**334. Finding geometric means.**

Find 3 geometric means between 24 and 15,000.

Including the 3 means and 2 extremes there will be 5 terms in all, of which $a_1 = 24$ and $a_5 = 15{,}000$.

Then $a_1 = 24$, (1) $a_1 \times r^4 = 15{,}000$. (2)

Dividing (2) by (1), $r^4 = 625$. $\therefore r = \sqrt[4]{625} = 5$.

Therefore the G. P., showing the 3 means, is 24, [120, 600, 3000], 15,000.

1. Insert 1 geometric mean between 4 and 9.
2. Insert 1 geometric mean between $6\frac{1}{4}$ and $5\frac{4}{9}$.
3. Insert 2 geometric means between 5 and 320.
4. Insert 3 geometric means between 6 and 486.
5. Insert 7 geometric means between 2 and 13,122.
6. Insert 3 geometric means between $+1$ and $-1$.
7. Insert 5 geometric means between $\frac{16}{27}$ and $6\frac{3}{4}$.

**335. Finding the sum of $n$ terms of a G. P.**

By definition and notation (§§ 331, 323), we have

$$s = a + ar + ar^2 \cdots ar^{n-1}, \qquad (1)$$

$$rs = \quad ar + ar^2 \cdots ar^{n-1} + ar^n. \qquad (2)$$

(2) is obtained by multiplying (1) by $r$.

(2) $-$ (1) $\qquad (r-1)s = ar^n - a.$

$\therefore$ $$s = \frac{ar^n - a}{r-1} \text{ or } \frac{rl - a}{r-1}. \qquad (C)$$

1. In a certain G. P. $a = 1$, $r = 2$, and $l = 1024$. Find $s$.
2. Find the sum of 9 terms of 1, 3, 9, ….
3. Find the sum of 7 terms of 8, 20, 50, ….
4. Find the sum of 6 terms of $1\frac{3}{5}$, $2\frac{2}{5}$, $4\frac{4}{5}$, ….
5. Find the sum of 10 terms of $2 - 2^2 + 2^3$, ….
6. Find the sum of $n$ terms of $3 + 2 + \frac{4}{3}$, ….
7. Find the sum of $n$ terms of $\frac{1}{3}$, $\frac{1}{2}$, $\frac{3}{4}$, ….

**336. Finding the sum of an infinite G. P.**

A **decreasing G. P.** is one in which $r < 1$, numerically, and in which, therefore, the terms become smaller and smaller as we pass from left to right in the series.

From formula (C) $s = \frac{a - lr}{1 - r} = \frac{a}{1 - r} - \frac{lr}{1 - r}$, in which $l = ar^{n-1}$.

Since $r < 1$, as $n$ becomes larger and larger $l$ becomes smaller and smaller, so that by taking $n$ large enough $s$ can be made to differ as little as we please from $\frac{a}{1 - r}$. This is expressed by saying:

($a$) The **limit** of $s$ as $n$ approaches infinity is $\frac{a}{1 - r}$.

($b$) The **sum** of the infinite G. P. $a + ar + ar^2 \cdots$, when $r < 1$, is

$$s = \frac{a}{1 - r}. \qquad (D)$$

Find the sum of 4, 2, 1, $\frac{1}{2}$, $\cdots$ to infinity.

$$s = \frac{a}{1 - r} = \frac{4}{1 - \frac{1}{2}} = 8. \quad Ans.$$

1. Find the sum of $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{12}$, $\cdots$ to infinity.

Find the following extended without limit:

2. $5 - \frac{1}{2} + \frac{1}{20} \cdots$.
3. $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$.
4. $\frac{3}{2} - \frac{2}{3} + \frac{8}{27} \cdots$.
5. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$.
6. $\frac{1}{4} - \frac{1}{16} + \frac{1}{64} \cdots$.
7. $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$.
8. $1 + \frac{1}{n} + \frac{1}{n^2} \cdots$.
9. $1 - \frac{1}{n} + \frac{1}{n^2} - \cdots$.
10. .757575 $\cdots$, or $\frac{75}{100} + \frac{75}{10000} + \frac{75}{1000000} \cdots$.
11. .878787 $\cdots$.
12. .365365365 $\cdots$.
13. .52727 $\cdots$.
14. .17285285 $\cdots$.

**337.** In the two equations

$$l = ar^{n-1} \cdots, \qquad (A) \qquad s = \frac{rl - a}{r - 1}, \qquad (C)$$

there are five letters, $a$, $r$, $l$, $n$, $s$. If any three of these are given, the equations may, in general, be solved for the other two.

1. In a certain G. P. $r = 3$, $n = 6$, $s = 728$. Find $a$ and $l$.

Substituting these values for $r$, $n$, and $s$ in (A) and (C), we have, respectively,

$$l = 243\,a \cdots, \qquad (1) \qquad 728 = \frac{3\,l - a}{2} \cdots. \qquad (2)$$

From (2), $3\,l - a = 1456.$

$\therefore$ $729\,a - a = 1456$, $a = 2$, $l = 486$.

2. How many terms of the G. P. 3, 6, 12, $\cdots$ must be taken so that their sum shall be 765?

Here, $a = 3$, $r = 2$, $s = 765$. For these values (A) and (C) are

$$l = 3 \times 2^{n-1} \cdots. \qquad (1) \qquad 765 = 2\,l - 3 \cdots. \qquad (2)$$

From (2), $l = 384$; then from (1), $2^{n-1} = 128$. $\therefore n = 8$.

1. Given $a = 1$, $l = 6561$, and $s = 9841$. Find $n$ and $r$.

2. Given $a = 2$, $l = 512$, and $r = 2$. Find $n$ and $s$.

3. What must be the value of $x$ so that $2x + 9$, $4x + 11$, and $8x + 9$ shall be in G. P.?

HINT. $(4x + 11) \div (2x + 9) = (8x + 9) \div (4x + 11)$. Why?

4. The sum of three numbers in A. P. is 12, and if the first is increased by 1, the second by 2, and the third by 4, the numbers will be in G. P. Find the numbers.

HINT. Let $x - y$, $x$, $x + y =$ the numbers.

Then, (1) $x - y + x + x + y = 12$. $\therefore x = 4$.

Therefore the numbers are $4 - y$, $4$, $4 + y$.

(2) $$\frac{4 + 2}{4 - y + 1} = \frac{4 + y + 4}{4 + 2}, \text{ or } \frac{6}{5 - y} = \frac{8 + y}{6}.$$

5. Find the first term of a geometric progression when the last term is 1536, the ratio 2, and the number of terms 10.

6. The last term of a geometric progression is 12,500, the first term 4, and the number of terms 6. Find the ratio and sum of the terms.

7. The last term of a geometric progression is 1536, the first term 3, and the sum of the terms 3069. Find the ratio.

8. The last term of a geometric progression is 885,735, the ratio 3, and the sum of the terms 1,328,600. Find the first term.

9. The first term of a geometric progression is 1, the ratio 3, and the sum of the terms 9841. Find the last term and the number of terms.

10. The first two terms of a series in geometric progression are $\frac{2}{3}$ and $\frac{1}{2}$. What are the next two terms?

11. The first and third terms of a geometric series are 4 and 16. What is the fourth term?

12. Suppose a body to move 20 mi. the first minute, 19 mi. the second, $18\frac{1}{20}$ mi. the third, and so on forever. Required the utmost distance it can reach.

13. A father divided $147 among three sons, the shares being in geometric progression; and the first had $63 more than the last. How much had each?

14. Find the geometric series in which the sum of the first two terms is 36, and of the next two 324.

15. A person who saved every year $1\frac{1}{2}$ times as much as he saved the previous year had in 7 years saved £102 19*s*. How much did he save the first year?

## HARMONIC PROGRESSION

**338.** A **harmonic progression** (H. P.) is a series whose terms are the reciprocals of the corresponding terms of an A. P.

Thus, $\frac{1}{5}, \frac{1}{9}, \frac{1}{13}, \cdots$ is a H. P. because the reciprocals of the terms, 5, 9, 13, $\cdots$ form an A. P.

The name *harmonic* is given to such a series because musical strings of uniform size and tension whose lengths are the reciprocals of 1, 2, 3, $\cdots$ vibrate in harmony.

Evidently all questions pertaining to H. P. can be answered by first converting it into an A. P.

Thus, to find the $n$th term of the H. P. $\frac{1}{7}, \frac{1}{11}, \frac{1}{4}, \cdots$ we find the $n$th term of the A. P. $7, 11, 4, \cdots$, and take its reciprocal.

Again, to find 5 harmonic means between 3 and 6, we find 5 arithmetic means between $\frac{1}{3}$ and $\frac{1}{6}$, and take their reciprocals.

There is one exception to this rule, viz. there is no formula for finding the sum of $n$ terms of a H. P.

1. The first two terms of a H. P, are 4 and 2; find the third term.
2. In a H. P. $a_1 = 8$, and $a_2 = 5$; find $a_{10}$.
3. In a H. P. $a_5 = \frac{1}{16}$, and $a_8 = \frac{1}{23}$; find $a_{15}$.
4. Find the 23d term of the H. P. $\frac{5}{8}, \frac{3}{2}, \cdots$.
5. Insert 2 harmonic means between 3 and 2.
6. Insert 6 harmonic means between 3 and $-1$.
7. Insert 5 harmonic means between $-1$ and $+1$.

Find the two numbers whose:

8. Arithmetic mean is 5 and harmonic mean $3\frac{1}{5}$.
9. Geometric mean is 20 and harmonic mean 16.

### ARITHMETIC, GEOMETRIC, AND HARMONIC MEANS

**339.** Given $m$ and $n$, any two positive real numbers, and $a$, $g$, and $h$, the arithmetic, geometric, and harmonic means of $m$ and $n$, respectively.

Prove the following:

1. $a = \frac{m+n}{2}$. 2. $g = \sqrt{mn}$. 3. $h = \frac{2mn}{m+n}$.

4. $a$ is greater than $g$. p. 149, (O).

5. $h$ is less than $g$. p. 149, (O).

6. The geometric mean of $a$ and $h$ is the same as the geometric mean of $m$ and $n$.

Therefore the square root of any imperfect square may be found to any desired degree of accuracy by taking successively the arithmetic and harmonic means of any two factors of the number.

Thus, to find $\sqrt{3}$, we take its two factors, say 1 and 3, and proceed as follows:

The arithmetic and harmonic means:
(1) of 3 and 1 are 2 and $\frac{3}{2}$;
(2) of 2 and $\frac{3}{2}$ are $\frac{7}{4}$ and $1\frac{2}{7}$;
(3) of $\frac{7}{4}$ and $1\frac{2}{7}$ are $\frac{97}{56}$ and $1\frac{68}{97}$.

. . . . . .

Now, as the process is continued, the arithmetic mean decreases and the harmonic mean increases; and therefore each approaches the geometric mean, which always lies between the two. Hence,

$\sqrt{3}$ differs from $\frac{3}{2}$ by less than $2 - \frac{3}{2}$ or $\frac{1}{2}$;

$\sqrt{3}$ differs from $1\frac{2}{7}$ by less than $\frac{7}{4} - 1\frac{2}{7}$ or $\frac{1}{49}$;

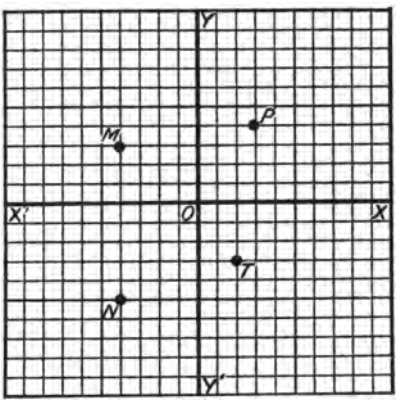
$\sqrt{3}$ differs from $1\frac{68}{97}$ by less than $\frac{97}{56} - 1\frac{68}{97}$ or $\frac{1}{5432}$.

. . . . . .

## XVI. GRAPHICAL METHOD

### AXES AND COORDINATES

**340. Graph paper** is paper ruled very accurately into small squares. It is manufactured in large quantities, and is not expensive. Paper of this kind furnishes an easy and satisfactory means of drawing the plot or graph of related variables, and of solving simultaneous equations involving two unknown numbers.



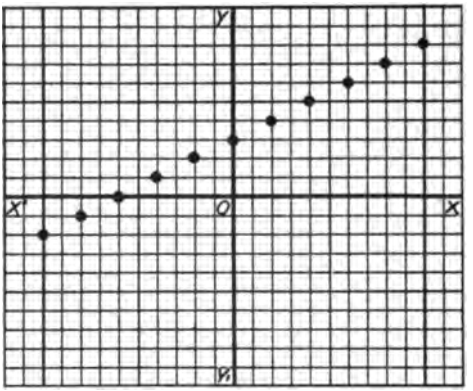

**341.** The **axes** are two lines of reference, $XX'$ and $YY'$, which are perpendicular to each other. $XX'$ is called the ***x*-axis**, $YY'$ the ***y*-axis**, and their point of intersection, $O$, the **origin.**

**342.** The **coördinates** of a point are the two numbers that locate the point.

The coördinates of the point $P$ are (3, 4), which means that the point is 3 units to the *right* of $YY'$ and 4 units *above* $XX'$.

Similarly, the point $N$ is $(-4, -5)$, being 4 units to the *left* of $YY'$, and 5 units *below* $XX'$. In general, $(a, b)$ is a point which is $a$ units to the right or left of $YY'$ according as $a$ is $+$ or $-$, and $b$ units above or below $XX'$ according as $b$ is $+$ or $-$.

**343.** When a point is represented by two numbers, as $(a, b)$, the first, $a$, is called the **abscissa** of the point, and the second, $b$, the **ordinate** of the point.




Let the student give the coördinates of each of the several points in this diagram, beginning with the point on the left, and taking the points in order.

Thus, $(-10, -2)$, $(-8, -1)$, and so on.

On the same diagram, or on another properly ruled paper, let the student *dot* or *plot* the following points:

| | | |
|---|---|---|
| 1. $(3, 5)$. | 2. $(3, -5)$. | 3. $(-3, +6)$. |
| 4. $(2, 7)$. | 5. $(-2, 4)$. | 6. $(-7, -9)$. |
| 7. $(8, 0)$. | 8. $(-5, 0)$. | 9. $(-2, -8)$. |
| 10. $(0, 5)$. | 11. $(8, -7)$. | 12. $(-5, +7)$. |
| 13. $(1, 6)$. | 14. $(4, -4)$. | 15. $(+8, -1)$. |

**344.** In general, the abscissa of a point is represented by $x$ and the ordinate by $y$. Thus $(x, y)$ is *any* point and $(4, -7)$ is a *particular* point in which $x = 4$ and $y = -7$.

Name the values of $x$ and $y$ in each of the particular points given above.

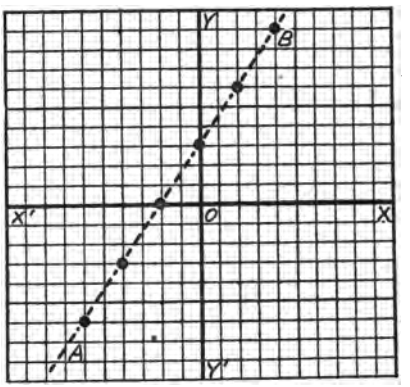
**345. The graph (or picture) of a linear equation.**

Take the equation, $2y - 3x = 6$. (1)

If in this equation we make $x = -6, -4, -2, 0, +2, +4$, the corresponding values of $y$ are $-6, -3, 0, 3, 6, 9$.

That is, (1) is satisfied by the following pairs of values of $x$ and $y$ (§ 46):

$(-6, -6), (-4, -3), (-2, 0)$,
$(0, 3), (2, 6), (4, 9)$.




If we consider these number-pairs the coördinates of points and dot the points, we shall find that they are all in a straight line.

Moreover, the coördinates of every point in the line $AB$, and those of no other points whatever, satisfy the given equation.

For these reasons the line drawn through these points is called the **graph** of the equation $2y - 3x = 6$.

An indefinite number of points might have been obtained by assigning other values, integral and fractional, to $x$, and finding the corresponding values of $y$.

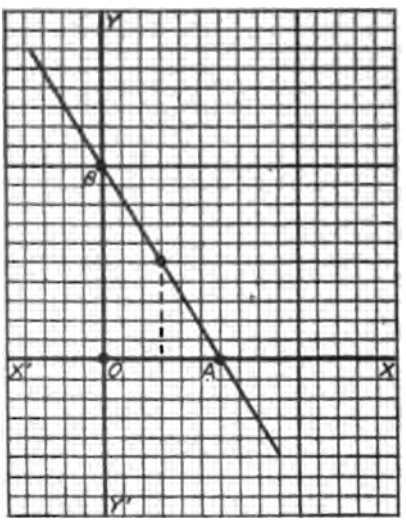
Plot the graphs of the following:

| | | |
|---|---|---|
| **1.** $y = x$. | **2.** $y + 2x = 8$. | **3.** $3y + 4x = 12$. |
| **4.** $y = 2x$. | **5.** $y - 3x = 9$. | **6.** $2y - 3x = 18$. |
| **7.** $y = \frac{1}{2}x$. | **8.** $y - 4x = -12$. | **9.** $2x + 5x = -10$. |
| **10.** $y = 3x$. | **11.** $y - 2x = -10$. | **12.** $4y - 7x = -56$. |
| **13.** $y = \frac{1}{3}x$. | **14.** $2y + x = -14$. | **15.** $3y + 5x = -15$. |

**346. Plotting the graph by a shorter method.**

It can be proved that the graph of every equation of the first degree in two variables is a straight line. Such equations are therefore called **linear equations.**

Hence, the graph of a linear equation may be plotted by drawing a straight line through any two of its points.




Thus, to draw the graph of $3y + 5x = 30$ we take any two values for $x$, as 0 and 3, and find the corresponding values of $y$, 10 and 5.

Then, (0, 10) and (3, 5) are two points of the graph, and by drawing a straight line through these points we have the graph required.

The above method applies to any linear equation, but when the graph does not pass through the origin the two points most easily determined are those where the graph cuts the axes. Hence,

(1) Find the point where the graph cuts the $x$-axis by making $y = 0$, and finding $x$. (2) Find the point where the graph cuts the $y$-axis by making $x = 0$, and finding $y$.

**347.** The ***x*-intercept** of a graph is that part of the $x$-axis lying between the origin and the graph; as $OA$.

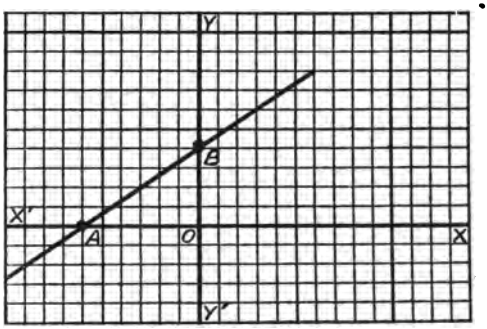
The ***y*-intercept** of a graph is that part of the $y$-axis lying between the origin and the graph; as $OB$.

1. Plot the graph of $3y - 2x = 12$.

For $y = 0$, $x = -6$, which is the $x$-intercept, $OA$.

For $x = 0$, $y = 4$, which is the $y$-intercept, $OB$.

Now draw a straight line through $A$ and $B$, and it is the graph required.




The graph of $x = 5$ is a straight line parallel to the $y$-axis and at a distance of 5 to the right of it. The graph of $y = -3$ is a straight line parallel to the $x$-axis and at a distance of 3 below it.

2. Plot the graph of $2y + 3x = 0$.

The values of $x$ may always be so selected as to render the value of $y$ integral. For $x = 0$, $y = 0$, for $x = 2$, $y = -3$.

Plot the graphs of the following:

| | | | |
|---|---|---|---|
| 1. | $y + 3x = 12$. | 2. | $4y + 3x + 12 = 0$. |
| 3. | $y - 2x = 10$. | 4. | $6y - 5x + 30 = 0$. |
| 5. | $4y + x = 16$. | 6. | $3y + 4x - 18 = 0$. |
| 7. | $5y + x = 15$. | 8. | $8y - 5x - 20 = 0$. |
| 9. | $2y + 3x = 6$. | 10. | $3y + 2x - 11 = 0$. |
| 11. | $3y - 2x = 9$. | 12. | $4y - 3x + 14 = 0$. |

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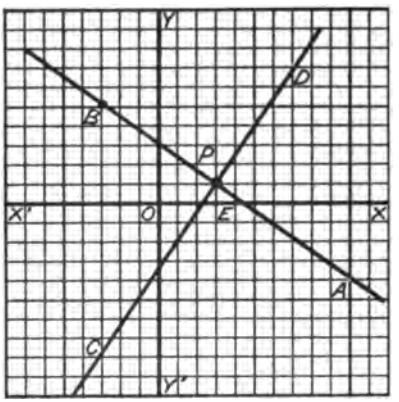
| | | | | | |
|---|---|---|---|---|---|
| 13. | $y - 6 = 0$. | 14. | $2y + 3x = 0$. | 15. | $2y = 5x$. |
| 16. | $x + 4 = 0$. | 17. | $4y - 5x = 0$ | 18. | $3y = 7x$. |
| 19. | $y + 7 = 0$. | 20. | $3y + 4x = 0$. | 21. | $y = -6$. |
| 22. | $y - 2 = 0$. | 23. | $5y - 2x = 0$. | 24. | $x = +5$. |

**348. Solving simultaneous linear equations by graphs.**

Solve graphically $\begin{cases} 2x+3y=9, & (1) \\ 3x-2y=7. & (2) \end{cases}$




Draw the graph of (1); that is, the line $AB$.

Draw the graph of (2); that is, the line $CD$.

It will be seen that the graphs intersect at the point $P$, (3, 1).

This being the only point in common to the two graphs, it is evident that $x=3$ and $y=1$ are the only values of $x$ and $y$ that will satisfy the given equations. Hence, the measurements of $OE$ and $EP$, the coördinates of the point of intersection, are the required values of $x$ and $y$.

Solve the following graphically, and check by solving the examples as in § 48:

1. $\begin{cases} 2x+y=9, \\ x+2y=8. \end{cases}$

2. $\begin{cases} 2x+y=13, \\ 3x-4y=3. \end{cases}$

3. $\begin{cases} x+4y=11, \\ 5x-2y=11. \end{cases}$

4. $\begin{cases} 7x-4y=81, \\ 5x-3y=57. \end{cases}$

5. $\begin{cases} x+6y=13, \\ 5x+2y=9. \end{cases}$

6. $\begin{cases} 3x+2y=30, \\ 2x+3y=30. \end{cases}$

7. $\begin{cases} 2x-3y=-7, \\ 4x-5y=-9. \end{cases}$

8. $\begin{cases} 3x+2y=42, \\ 3y-4x=12. \end{cases}$

9. $\begin{cases} 8x+3y=12, \\ 12x+5y=16. \end{cases}$

10. $\begin{cases} 5x-3x=36, \\ 7x-5y=56. \end{cases}$

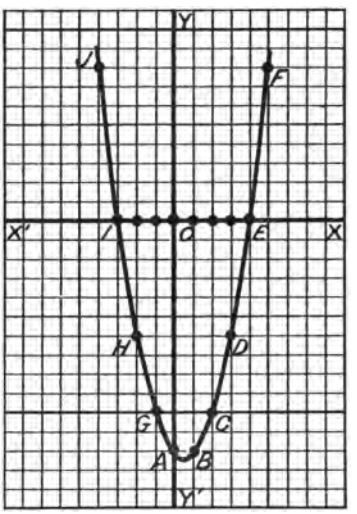
## QUADRATIC EQUATIONS

**349. Plotting the graphs of quadratics in two variables.**

Plot the graph of $y = x^2 - x - 12$.

As before, we substitute 0, 1, 2, ⋯, also $-1$, $-2$, ⋯, for $x$, dot the points of the graph thus determined, and draw a smooth curve through the dots, as follows:

| $x$ | $y$ | Points |
|---|---|---|
| 0 | $-12$ | $A$ |
| 1 | $-12$ | $B$ |
| 2 | $-10$ | $C$ |
| 3 | $-6$ | $D$ |
| 4 | 0 | $E$ |
| 5 | 8 | $F$ |
| ⋮ | ⋮ | ⋮ |
| $-1$ | $-10$ | $G$ |
| $-2$ | $-6$ | $H$ |
| $-3$ | 0 | $I$ |
| $-4$ | 8 | $J$ |
| ⋮ | ⋮ | |



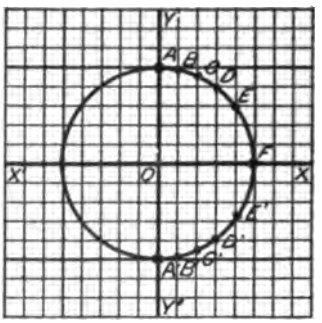

Note that the roots of $x^2 - x - 12 = 0$, viz., 4 and $-3$, are the $x$-intercepts of the graph of $y = x^2 - x - 12$.

**350.** If $f(x) = 0$ stands for any equation in $x$, the roots of $f(x) = 0$ are the $x$-intercepts of the graph of $y = f(x)$.

Plot the graphs of the following:

1. $y = x^2 - 4x$.
2. $y = x^2 - 16$.
3. $y = x^2 - 8x + 12$.
4. $y = x^2 + 3x$.
5. $y = x^2 - 25$.
6. $y = x^2 + 9x + 18$.
7. $y = x^2 - 5x$.
8. $y = x^2 - 81$.
9. $y = x^2 - 2x - 15$.

1. Construct the graph of $y^2 + x^2 = 25$.




Solve for $y$,

$$y = \pm\sqrt{25 - x^2}.$$

| $x$ | $y$ | Points |
|---|---|---|
| 0 | $\pm$ 5 | $A$, $A_1$ |
| 1 | $\pm$ 4.9 | $B$, $B_1$ |
| 2 | $\pm$ 4.6 | $C$, $C_1$ |
| 3 | $\pm$ 4 | $D$, $D_1$ |
| 4 | $\pm$ 3 | $E$, $E_1$ |
| 5 | 0 | $F$ |
| 6 | $\pm\sqrt{-11}$ | |

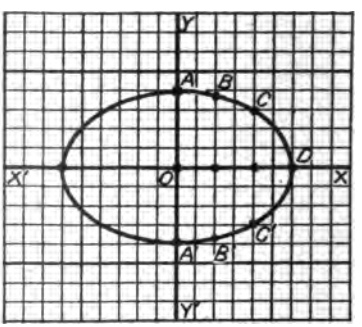
The same values for $y$ are obtained for negative values of $x$, respectively. Since any value of $x$ greater than $+5$ or less than $-5$ renders $y$ imaginary, the graph does not extend further than 5 units to the right or left of the origin. The graph is evidently a circle.

2. Construct the graph of $9y^2 + 4x^2 = 36$.

Solve for $y$,

$$y = \pm\tfrac{2}{3}\sqrt{9 - x^2}.$$

| $x$ | $y$ | Points |
|---|---|---|
| 0 | $\pm$ 2 | $A$ and $A_1$ |
| 1 | $\pm$ 1.9 | $B$ and $B_1$ |
| 2 | $\pm$ 1.5 | $C$ and $C_1$ |
| 3 | 0 | $D$ |

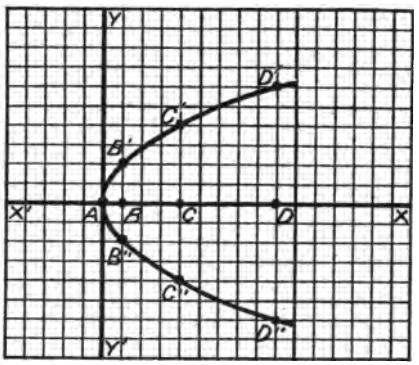


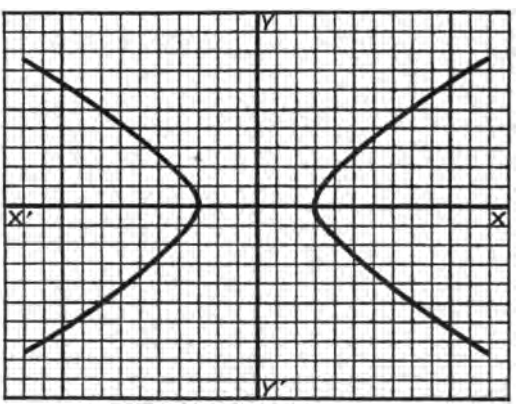

Evidently the values of $y$ for negative values of $x$ are the same as for positive values of $x$. Hence, the graph, which is an ellipse, is the same on the left of $YY'$ as on the right of $YY'$.

3. Show that the figure on the right is the graph of

$$y^2 = 4x.$$

This graph is a **parabola.**



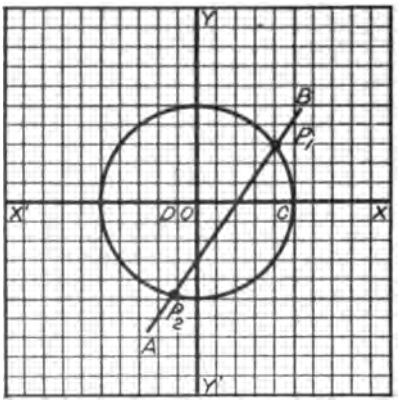

4. Show that the figure above is the graph of

$$9y^2 - 4x^2 = -36.$$

This graph is a **hyperbola.**

5. Show that one branch of $xy = 12$ lies above $XX'$ and to the right of $YY'$, and the other branch below $XX'$ and to the left of $YY'$.

**351. Solving quadratics by graphs.**



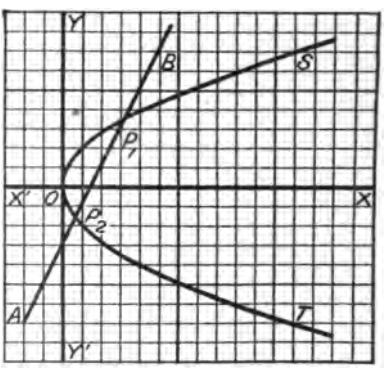

1. Solve $\begin{cases} y^2 + x^2 = 25, \\ 2y - 3x = -6. \end{cases}$

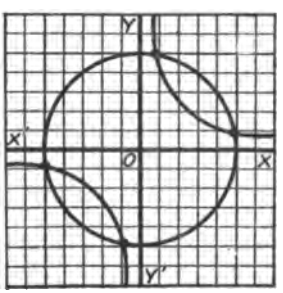
Draw the graphs of the equations, intersecting at $P_1$ and $P_2$. The measurements of $OC$ and $CP_1$ (4, 3), and of $OD$ and $DP_2$ $(-1\frac{8}{13}, -4\frac{1}{13})$, are the values of $x$ and $y$.

Check the results by solving the equation.

2. Solve $\begin{cases} y^2 = 4x, \\ y - 2x = -3. \end{cases}$

The graphs, $SOT$ and $AB$, intersect at $P_1$ and $P_2$. The measurements of the linear coördinates of $P_1$ are, approximately, (3.4, 3.7); and of $P^2$, (.65, −1.6), which are approximately the pairs of values of $x$ and $y$.




3. Solve $\begin{cases} y^2 + x^2 = 25, \\ 2xy = 9. \end{cases}$

The graphs intersect in four points, and the measurements of the linear coördinates of these points are, about:

(4.9, .9), (.9, 4.9), (−.9, −4.9), (−4.9, −.9). *Ans.*

### PROBLEMS

Solve graphically and check by ordinary solution:

1. $\begin{cases} y^2 + x^2 = 36, \\ 4y - 3x = 0. \end{cases}$

2. $\begin{cases} y^2 + x^2 = 100, \\ 7y - 12x = 0. \end{cases}$

3. $\begin{cases} xy = 10, \\ y - x = 3. \end{cases}$

4. $\begin{cases} y^2 = 2x, \\ 8y = x. \end{cases}$

5. $\begin{cases} y = x^2 + 9, \\ y = x^2 - 5x + 6. \end{cases}$

6. $\begin{cases} y^2 + x^2 = 25, \\ 9y^2 + 4x^2 = 144. \end{cases}$

7. $\begin{cases} 4y^2 + x^2 = 36, \\ xy = 4. \end{cases}$

8. $\begin{cases} 4x^2 - 9y^3 = 36, \\ x^2 + y^2 = 25. \end{cases}$

For additional practice the student can solve such of the examples on page 241 as he or the teacher may select.

**352.** Eliminating $y$ between $ay + bx = c$ and $y = x^2$, we obtain $ax^2 + bx = c$; hence the roots of any quadratic, like $3x^2 + 2x = 5$, may be found graphically by finding the abscissas of the points of intersection of the straight line $3y + 2x = 5$ and the simple curve $y = x^2$.

**353.** It was stated on page 229 that the equation

$$\frac{x-2}{x-5} - \frac{3}{x-2} = \frac{9}{(x-2)(x-5)}$$

has no roots. Let the student prove this by showing that the graph of

$$y = \frac{x-2}{x-5} - \frac{3}{x-2} - \frac{9}{(x-2)(x-5)}$$

does not cut the $x$-axis, and therefore has no $x$-intercept, §§ 347, 350.

### FEEDS AND FEEDING

**354.** The chief nutrients required by a person or an animal are, **Digestible Protein**; **Digestible Carbohydrates**; and **Digestible Fat.**

Protein is represented by the white of eggs, the gluten of wheat and other grains, the fibrin of meat, the casein of milk, etc.

Carbohydrates are represented by the starches and sugars.

Fat is represented by animal and vegetable fats and oils.

The function of protein is to build up and repair such organs and tissues as the muscles, tendons, internal organs, the blood, the skin, etc., and also assists largely in the formation of milk.

The function of the carbohydrates is to undergo oxidation and produce heat and energy.

The function of fats is similar to that of the carbohydrates, viz., to produce heat and energy, but to a somewhat greater extent.

**355.** The following is a table of digestible nutrients in certain feeds:

| Feeds | Per Cent of Protein | Per Cent of Carbohydrates | Per Cent of Fats |
|---|---|---|---|
| Alfalfa Hay | 10.6 | 28.2 | .9 |
| Butter | 1.0 | 0 | 85.0 |
| Corn (Grain) | 7.8 | 65.5 | 4.3 |
| Corn Fodder (Leaves) | 5.3 | 26.3 | 2.0 |
| Cotton-seed Meal | 37.2 | 15.1 | 12.2 |
| Eggs | 15.0 | 0 | 10.0 |
| Timothy Hay | 2.8 | 28.3 | 1.4 |
| Oats (Grain) | 9.3 | 44.8 | 3.5 |
| Cooked Beef | 27.5 | 0 | 15.5 |
| Corn Bread | 7.9 | 46.3 | 4.7 |
| Fat Pork | 14.5 | 0 | 37.3 |
| Irish Potatoes | 1.9 | 22.0 | 0.2 |
| Light Bread | 6.8 | 52.3 | .7 |
| Milk | 3.0 | 5.0 | 4.0 |

## NORMAL RATIONS

**356.** A **ration** is the feed given to an animal during 24 hours.

**357.** A **normal ration** is the ration which is adapted, both as to quantity and ingredients, largely to the taste, and especially to the needs, of the animal.

It has been found that normal rations, even for animals of the same kind, vary in amounts and in the proportions of protein, carbohydrates, and fats, according to the growths of animals and the purposes for which they are fed.

**358.** The following table shows the amounts of digestible nutrients in normal rations, calculated for 1000 pounds live weight.

| Kind of Animal | Digestive Nutrients in Pounds | | |
|---|---|---|---|
| | Protein | Carbohydrates | Fats |
| Fattening cattle (2d period) . . . | 3.0 | 14.5 | .7 |
| Horse (heavy work) . . . . . | 2.3 | 12.5 | .8 |
| Horse (light work) . . . . . | 1.5 | 9.5 | .4 |
| Dairy cow (2 gal. milk daily) . . | 2.0 | 11.0 | .4 |
| Dairy cow (2¾ gal. milk daily) . . | 2.5 | 13.0 | .5 |
| Fattening swine (2d period) . . . | 4.0 | 24.0 | .5 |
| Ox (heavy work) . . . . . | 2.8 | 13.0 | .8 |
| Man (doing active work) . . . . | ⅓, or .33 | 1.1 | ⅓, or .33 |
| Maintenance requirements for cows per 100 lb. live weight . . . | .07 | .7 | .01 |

### NUTRITIVE RATIO

**359.** A term of importance, used in feeding parlance, is **nutritive ratio.** This means the ratio of the total digestible carbohydrates to the digestible protein.

The total amount of carbohydrates is obtained by adding to the digestible carbohydrates the product of the digestible fats by $2\frac{1}{4}$, because 1 lb. of fat produces as much heat and energy in the body as $2\frac{1}{4}$ lb. of carbohydrates.

Thus, the total amount of digestible carbohydrates in corn is $65.5\% + 2\frac{1}{4} \times 4.3\% = 75.2\%$; and therefore the nutritive ratio is $75.2 \div 7.8$, or 9.6.

**360.** A ration, or feed, is **narrow,** or **wide,** according as the nutritive ratio is small or large. Any nutritive ratio between 5.5 and 8 may be considered a medium ratio.

Find the nutritive ratio of the following feeds:

1. Corn fodder.
2. Cotton-seed meal.
3. Light bread.
4. Cooked beef.
5. Fat pork.
6. Timothy hay.

Find the nutritive ratio of the normal ration for:

7. Fattening cattle.
8. Horse (light work).
9. Ox (heavy work).
10. Horse (heavy work).
11. Fattening swine.
12. Dairy cow (giving 2 gal. milk daily).
13. Dairy cow (giving $2\frac{3}{4}$ gal. milk daily).

14. What ought the per cent of fat in corn bread be in order that the nutritive ratio should be 9?

### BALANCING RATIONS

**361. Balancing rations** is finding what amounts of different kinds of feed must be mixed so as to produce a normal ration.

NOTE. Such of the following problems as involve equations of two unknown numbers may be solved graphically, at the discretion of the teacher. It will be sufficiently accurate to find the required amounts to the nearest *tenth* of a pound or ounce. Hence, if ordinary graph paper is used, make the plot on the scale of 5 or 10 *linear divisions* to the *unit*.

### PROBLEMS

1. What amounts of alfalfa hay and corn should be given daily to an ox doing hard work to obtain the necessary amounts of protein and carbohydrates?

(1) The table on page 291 shows that the ox needs daily 2.8 lb. protein and 13 lb. carbohydrates.

(2) The table on page 290 shows that of alfalfa hay 10.6% or .106 is protein and 28.2% or .282 is carbohydrates; and of corn 7.8% or .078 is protein and 65.5% or .655 is carbohydrates.

Therefore, if we let $x$ and $y$ = the necessary numbers of pounds of the hay and corn, respectively, we have

$$.106\,x + .078\,y = 2.8,$$
$$.282\,x + .655\,y = 13,$$

or, multiplying by 100,

$$10.6\,x + 7.8\,y = 280,$$
$$28.2\,x + 65.5\,y = 1300.$$

Whence, $x = 17.3,\ y = 12.4.$

Evidently, when two kinds of feeds are mixed to produce a normal ration, one of the feeds must have a wider, and the other a narrower, nutritive ratio than that of the normal ration.

2. How many pounds each of corn and alfalfa hay are needed daily to contain the right amounts of protein and fat for a hard-worked horse?

NOTE. In problems like these, a negative answer indicates the impossibility of fulfilling the requirements.

3. How many ounces each of butter and light bread are required per day for a workingman in order to supply him with normal amounts of protein and fats?

4. If a workingman's ration consists of corn bread and fat pork, how many ounces of each should he eat per day to get the requisite amounts of protein and fats?

5. How much of eggs and cooked beef will form a normal ration for a man doing active muscular work?

6. How much each of oats, fodder, and alfalfa hay will form a normal ration for an ox weighing about 1000 lb. and doing heavy work?

7. I have a number of milk cows whose aggregate live weight is 10,000 lb. If each gives about $2\frac{3}{4}$ gal. milk daily, how much each of corn, cotton-seed meal, and timothy hay would form a normal ration for all of them together?

8. To contain a daily ratio of 3.1 oz. of protein and 18 oz. of carbohydrates, how many ounces each are needed of milk and light bread?

# ANSWERS

NOTE. While the answers to most of the problems are given, it is not deemed necessary or advisable to give all of them. When two or more problems are embraced in the same number, as a rule the answer to the last only is given.

**Page 7.**—**2.** $6x$. **3.** $10a$. **4.** $2b$. **5.** $6c$. **6.** $3d$. **7.** 3. **8.** $25y$. **9.** $11y$. **10.** $200y$. **11.** $8y$. **12.** 7.

**Page 8.**—**1.** $7n$, $5n$. **2.** $10x$, $8x$. **3.** 0. **4.** 0. **5.** $2c$.

**Page 9.**—**1.** 4. **2.** 6. **3.** $x$. **4.** $y$. **5.** $n$. **6.** $b$. **7.** $a+n$. **8.** $a-b$. **9.** $x+3$. **10.** $y-6$. **11.** $m-2$. **12.** $c-5$. **13.** $x-3$. **14.** $(x-9)$ ft. **15.** $x-y$. **16.** $16x$. **17.** $7y$.

**Page 11.**—**1.** $7n$, 2. **2.** $9y$, 13. **3.** $5c$, 5. **4.** $6a$, 9. **5.** $13x$, 13.

**Page 12.**—**1.** 4. **2.** 3. **3.** 8. **4.** $c$. **5.** $6x$. **6.** $5a¢$. **7.** $3n¢$. **8.** $6x$ mi. **9.** $45x¢$. **10.** $\$24x$. **11.** $\$24y$. **12.** $3y¢$. **13.** $24¢$. **14.** $81x¢$. **15.** $2x+3$. **16.** $\$(12a-10)$.

**Page 13.**—**1.** 28. **2.** 2. **3.** 18 and 5. **4.** 11. **5.** 3. **6.** 23. **7.** 2. **8.** 3. **9.** 17. **10.** 12. **11.** 5. **12.** 33. **13.** 12.

**Page 14.**—**3.** 7. **4.** 10. **6.** 60. **8.** 9 **10.** 16. **12.** 50. **14.** 48. **16.** 21. **18.** $4+16=10+10$, or $20=20$. **20.** $17=17$.

**Page 15.**—**6.** 6. **8.** 7. **10.** 3. **12.** 4. **14.** 3. **16.** 4. **18.** 4. **19.** 6. **20.** 8.

**Page 16.**—**1.** 9, 27. **2.** 7, 35. **3.** 5, 40. **4.** 7, 28. **5.** 6, 66. **7.** 8, 24.

**Page 17.**—**8.** 8 yr., 40 yr. **9.** \$15, \$150. **10.** 15. **11.** 7 calves, 14 pigs. **12.** 4, 28. **13.** 9. **14.** $4¢$, $16¢$. **15.** 60. **16.** 8 yr., 16 yr., 48 yr. **17.** A, \$6; B, \$18; C, \$24.

**Page 18.**—**19.** 10, 22. **20.** 18, 13. **21.** 17, 18. **22.** 10, 11, 12. **23.** 11, 8. **24.** 9, 23. **25.** A, 22 yr.; B, 25 yr.; C, 31 yr. **26.** 11, 25. **27.** $45¢$, $55¢$, $70¢$.

**Page 20.**—**2.** $12x$. **4.** $14y$. **6.** $-a$. **8.** $5b$. **10.** $a-b$. **12.** $-b$. **13.** $10x+9$. **14.** $5a$. **15.** $-3$.

**Page 21.**—**1.** 9. **2.** $8x-3y$. **3.** $a+b-c$. **4.** $12a-3b+2c$. **5.** $c-d+x$. **6.** $4c-5d-2$. **7.** $13x$. **8.** $6a+4b-c$. **9.** $11n-2r-4s+3$. **10.** $4d+11e-f$. **11.** $-b+3c$.

**Page 22.** — 1. $\$11x$, $\$22$. 2. 340. 3. South, $26x$ yd. 4. South, 650 yd. 5. $n+1$, $n+2$. 6. 13, 14. 7. $(m+n+15)$ mi. 8. 53 mi. 9. $(a+b-35)¢$. 10. $105¢$. 11. $3x+26$. 12. 296. 13. $3x+18$.

**Page 24.** — 2. $-3$. 4. $-15$. 6. $13y$. 8. $-4b$. 10. $23x$. 12. $-30d$.

**Page 25.** — 2. $-10°$, $+3°$. 9. $8-a$. 10. $6+a$. 12. $-13a$.

**Page 26.** — 14. $5a-3c$. 15. $4x+9y$. 16. $-3a+9c-3d$. 17. $4n+4p-33$. 18. $6x-4$. 19. $2a+5b$. 20. 30.
1. $2x$. 2. $4a-b+6$. 3. $2y$. 4. $a-5b-1$. 6. $5c+6d$. 7. 4. 8. $-2x-8y-7$. 10. $2a+2b-3$. 11. $+1$. 12. 19. 13. $-4$; $+6$; $+3$; $-1$; $+1$.

**Page 27.** — 1. $\$(70-a)$. 2. $25-m$. 3. $n-1$, $n-2$. 4. $\$(8-a)$. 5. $\$(35-m-n)$. 6. $c-25$. 7. $\$(g-b)$. 8. $\$(m+a-50)$. 9. $(m+n-115)¢$. 10. $\$(6c-25)$.

**Page 29.** — 6. $-\$60$. 8. $+\$300$. 10. $+32n¢$. 12. $63x¢$. 14. $42a$ rd. 16. $28n$ rd.

**Page 30.** — 1. $12a+18b-12c$. 2. $15x-10y+50$. 4. $56c+35d-21x$. 6. $-72a+24n-56$. 8. $-30e+25f+40$. 11. $-15x$. 13. $-84a$. 15. $-35b$. 17. 0. 18. $25x-35$. 20. $42a-49b+28$.

**Page 31.** — 1. $4x$. 2. $33x$. 3. $\$(7m+8n)$. 4. $\$(5m-7n)$. 5. $4x+5$. 6. $4m-12$. 7. $\$48m$. 8. $\$(100-4n)$. 9. $\$(80+n)$. 10. $(280-8m)¢$. 11. $5n$ mi. 12. $100-8x-7y$.

**Page 32.** — 1. 11. 2. 6. 3. 8. 4. 7. 5. 9. 6. 6. 7. 5. 8. 8. 9. $6a$. 10. 25. 11. $9b$. 12. 28.

**Page 33.** — 1. 16. 2. 14. 3. $-19$. 4. $-14$. 5. $-12y$. 6. $-25x$. 7. $+\frac{3}{5}$. 8. $-\frac{5}{7}$. 9. $-\frac{2}{3}$. 10. $-\frac{5c}{3}$. 11. $-\frac{6x}{c}$.

**Page 34.** — 1. $4x-5$. 2. $6a-2b+15$. 4. $26c-15d-29x$. 6. $-19+12x+34$. 8. $6b+3c-78$. 10. $9x-13y+17$. 12. $8a-46-c$. 14. 7. 15. $-42x$. 17. 44. 19. $5a+42$. 20. $-6x-11y+19$. 21. $4a-11b+3$.

**Page 35.** — 1. $\frac{n}{4}¢$. 2. $\frac{a}{n}$. 3. $\frac{m}{2}$. 4. $\frac{a+b}{2}$. 5. $\frac{a+b+c}{3}$ lb. 6. $\frac{9m}{4}$ ft. 7. $\frac{40}{a}$. 8. $\frac{s-x}{5}$. 9. $\frac{4m+6n}{10}$ yr. 10. $\frac{3a+2b}{5}$. 11. $\frac{360-n}{8}$. 12. $\frac{500-10m}{m}$.

**Page 38.** — 1. 7. 2. 1. 4. 10. 5. $8n$. 6. 2. 7. $22m$. 8. 6. 9. $-5a$. 10. 17. 11. $-8b$. 12. 7. 14. $5a$. 16. $11a$. 18. $6b$. 19. $-3$. 20. 2. 21. $-7$. 22. $-5$. 24. 4. 25. 8. 26. 11. 28. $-7$. 30. 9. 32. 7.

**Page 39.** — 1. 2. 2. 8. 4. 19. 6. 2. 8. 4. 10. 6. 12. 4. 14. 3. 15. $2n$. 16. $-4$. 17. $3a$. 18. 10. 19. $3b$. 20. 3.

**Page 42.** — 1. $x=1$, $y=2$. 2. $x=5$, $y=1$. 3. $x=5$, $y=4$. 4. $x=12$, $y=9$. 5. $x=5$, $y=2$. 6. $x=20$, $y=15$. 7. $x=8$, $y=8$. 8. $x=3$, $y=5$. 9. $x=5$, $y=2$. 10. $x=2$, $y=1$. 11. $x=5$, $y=2$. 12. $x=10$, $y=3$. 13. $x=9$, $y=4$. 14. $x=2a$, $y=3a$. 15. $x=5$, $y=7$. 16. $x=10b$, $y=b$. 17. $x=8n$, $y=16n$. 18. $x=-3c$, $y=-2c$.

**Page 43.** — 1. 23, 11. 2. 19, 9. 3. 15, 7. 4. 14 oranges, 9 apples. 5. \$5 and \$8. 6. 5¢ and 15¢. 7. 8 and 6. 8. A, \$15; B, \$21. 9. Horse, \$168; buggy, \$75. 10. 8, 7.

**Page 44.** — 12. 16. 13. 42. 14. 83 and 38. 15. 36.

**Page 47.** — 1. 90. 2. 57. 3. 112. 4. 24. 5. 360. 6. 27. 7. 720. 8. 20. 9. 4320. 10. $-31$. 11. $-68$. 12. $-5$. 13. 200. 14. 0.

**Page 48.** — 1. 11. 2. 81. 3. 4. 4. 9. 5. 147. 6. 13. 7. 6. 8. 9. 9. 8. 10. 3. 11. 40. 12. 35. 13. 9. 14. 56. 15. 7. 16. 5. 17. $a+5b^2$, 23. 18. $2a-4c^3$, 2. 19. $ab(a-c)(b-d)$, 24. 20. $a^3+\frac{1}{2}(a+b+c)$, 30. 21. $3(a+b)+2(c-d)$, 13. 22. $a^3+b^3+c^3$, 36. 23. $\frac{a^2+6}{c}-b^2$, 11. 24. $(a+c)^2-(a^2+c^2)$, 6. 25. $(a+c)^3-(a^3+c^3)$, 36. 26. The square of the difference of $a$ and $c$ less the difference of their squares, $-4$. 27. The difference of the fourth powers of $a$ and $b$ divided by the difference of their squares, 13.

**Page 50.** — 1. $10a^2c$. 2. $-2xy^3$. 3. $5x^2z^2$. 4. $-3ab^2c^3$. 5. $3(ab)^2$. 6. $12(x-y)$. 7. $6ab$. 8. $5a^2c$. 9. $5mn$. 10. 0. 11. $-5cyz^2$. 12. $3a^2bc^3$. 13. $-86x^3y^3$. 14. $11(a^2-b)$. 15. $-6(a+b)^2$. 16. $17(a+b+c)$. 17. $-30(x+y+z)$. 18. $26(6x-7y)$.

**Page 51.** — 1. $14a^2b-4ab$. 2. $7a^2-a+6$. 3. $-3x^3+x^2+2x$. 4. $2a^3+2ab^2$. 5. $7x^2$. 6. $12x^3-2x^2+2x+5$. 7. $2a^2+2a+6$. 8. $8abcd-3bx-2m$. 9. 0. 10. $12-7ab+2ac$. 11. $23ax^3+10ay^3-15az^3$. 12. $6x^2+2y^2+5$. 13. $3a^2-2a+3$. 14. $7a^2c-4x$. 15. $2(a+b)+(c-x)+1$. 16. $4(a-x)+2ax+5$. 17. $(x+y)-5x^2y^2+x^2-7y^2$.

**Page 52.** — 1. $4x^2y^3$. 2. $11a^3b$. 3. $-15c^4x^2$. 4. $-4m^2n$. 5. $2x^2-3x+9$. 6. $3a^2-4ab+b^2$. 7. $6x^2y-4x^2-5xy^2-7xy-1$. 8. $x^3-x^2+7$. 9. $2a+2b+4c$. 10. $4a^3+3a^2-6a$. 11. $a^3-4a^2b+2ab^2-b^3$. 12. $8ax+d+y$. 13. $3ax+12(a-b)$. 14. $5(a+b)-9(x-y)$. 15. $a^2+4ab+ab^2-1$. 16. $10x^2+2xy+10$. 17. $a+b+c$. 18. $2x^2-3x-8$. 19. $a^2+8ab-x-y$.

**Page 53.** — 1. $10a^3b^2$. 2. $-24x^6y^3z$. 3. $-40a^5b^4c^2x$. 4. $42c^5d^7n^3x$. 5. $12a^3c$. 6. $72a^5b^7cd$. 7. $-6x^5y$. 8. $-21x^9y^7z^6$. 9. $-20a^6b$. 10. $10a^2(a-x)^3$. 11. $+a^7d^3$. 12. $-12x^4(a-b)^7$. 13. $-3x^3yz$. 14. $-24c^4d^5n^4y$. 15. $40a^{10}b^2c^3$. 16. $28m^3(n+3)^3$. 17. $48a^3(x+y)^7$. 18. $-30a^2b^2(a-b)^6$. 19. $-49a^3bc(n+x)^2$. 20. $30ax^5y^7z^2$. 21. $24(a+x)^9$.

**Page 54.** — 1. $5x^3 + 40x^2$. 2. $-9x^3y^3 + 18x^2y^4 - 24xy^5$. 3. $-2a^4 + 10a^5$. 4. $8a^4b^5 + 10a^3b^6 - 4a^2b^7$. 5. $-9x^2 + 12xy$. 6. $12n^6r^4 - 16n^5r^5 + 4n^4r^6$. 7. $-16xy + 20y^2$. 8. $-5c^6d^3 + 2c^5d^4 + 9c^4d^5$. 9. $x^5y^2 + x^2y^5$. 10. $42a^2n^7 - 28a^2n^5 + 56a^2n^3$. 11. $-x^5 + 4x^3y$. 12. $8x^4y - 6x^3y^2 - 12x^2y^3$. 13. $-x^7 - 3x^5y$. 14. $-4x^3y^3 + 8x^2y^2 - 16xy$. 16. $3a^4b^3 + 6a^3b^4 + 18ab^6$. 17. $x^5 + 4x^4y - 3x^2y^2$. 18. $6a^4b^4 - 10a^3b^5 - 16a^2b^6$. 19. $5a^3b^2 - 40a^2b^4 + 30ab^6$. 20. $a^2b^2xy + a^3bx^2y + abx^2y^2$. 21. $x^5y^5 - x^3y^7 + x^2y^9$. 22. $12m^3n - 16m^2n^2 - 2mn^3$. 23. $-12a^3xy^2 + 20ax^2y^3 - 8ay^4$. 24. $-5x^6 - 5x^5 - 5x^4 - 5x^3$. 25. $-6a^3ce + 3ac^3e - 2ace^3$. 26. $20(a+x)^3 + 12(a+x)^4$. 27. $-6ac^2(a-b)^3 + 8a^2c(a-b)^4$.

**Page 55.** — 1. $a^2 + 2ab + b^2$. 2. $x^4 - 4y^2$. 3. $a^2 - b^2$. 4. $x^4 + x^2y - 12xy^2$. 5. $6a^2 + ac - c^2$. 6. $x^5 + 2x^4 + x^3$. 7. $16x^2 + 4xy - 6y^2$. 8. $x^3 - 1$. 9. $x^4 - 16$. 10. $y^4 - 16$. 11. $6x^3y^2 - 13x^2y^3 + 21xy^4 - 8y^5$. 12. $a^5 - a^4b - 2a^3b^2 + 2a^2b^3 + ab^4 - b^5$. 14. $a^2 + 2ab + b^2 - c^2$. 16. $2a^2 - 3ax - ay - 2x^2 + 2xy$. 18. $6x^2 - 11xy - 12x - 10y^2 + 30y$. 20. $x^6 - 64$.

**Page 56.** — 1. $a^2 + 2ac + c^2$. 2. $x^2 + 2x + 1$. 3. $4x^2 + 16x + 16$. 4. $b^2 + 2bx + x^2$. 6. $4x^2 + 12x + 9$. 8. $y^2 + 14y + 49$. 10. $x^2 + 4x + 4$. 12. $16a^2 + 8ab + b^2$. 14. $c^2 + 2cy + y^2$. 16. $4x^2 + 12xy + 9y^2$. 17. $25x^4y^2 + 20x^3y^3 + 4x^2y^4$. 19. $9a^2b^2 + 30a^2bc + 25a^2c^2$. 21. $4c^2x^2 + 12cx^2y + 9x^4y^2$. 23. $16a^4 + 40a^3bc + 25a^2b^2c^2$. 25. $9a^4x^2 + 60a^3xy + 100a^2y^2$. 27. $900 + 120 + 4 = 1024$. 29. $100 + 160 + 64 = 324$. 31. $3600 + 240 + 4 = 3844$.

**Page 57.** — 1. $a^2 - 2ax + x^2$. 2. $x^2 - 6x + 9$. 3. $4x^2 - 20x + 25$. 4. $b^2 - 2by + y^2$. 6. $16x^2 - 24x + 9$. 8. $25 - 10y + y^2$. 10. $x^2 - 12x + 36$. 12. $9a^2 - 6ab + b^2$. 14. $x^2 - 18x + 81$. 16. $9x^2 - 12xy + 4y^2$. 17. $16a^4b^2 - 24a^3b^3 + 9a^2b^4$. 19. $25x^2y^4 - 100xy^3z + 100y^2z^2$. 21. $25a^2b^2 - 60a^2bcx + 36a^2c^2x^2$. 23. $9a^2x^2 - 12ax^3y + 4x^4y^2$. 25. $a^4x^2 - 24a^3x^3 + 144a^2x^4$. 27. $400 - 160 + 16 = 256$. 29. $1600 - 240 + 9 = 1369$. 31. $2500 - 200 + 4 = 2304$.

**Page 58.** — 2. $9x^2 - 4y^2$. 4. $25a^2 - 16b^2$. 6. $n^2x^2 - 25y^2$. 8. $x^4 - 36x^2$. 10. $81a^2 - m^2y^2$. 12. $9x^4y^2 - 16x^2y^4$. 14. $25a^2c^6 - 9b^2c^4$. 16. $16n^4x^2 - 64n^2y^6$. 18. $144x^8 - 100c^{10}$. 20. $900 - 49 = 851$. 22. $4900 - 9 = 4891$. 24. $6400 - 16 = 6384$. 26. $10{,}000 - 64 = 9936$. 28. $10{,}000 - 9 = 9991$.

**Page 59.** — 1. $x^2 + 11x + 28$. 2. $x^2 - 2x - 120$. 4. $x^2 + 2ax - 8a^2$. 6. $x^2 - 35x + 300$. 8. $x^2 - 2nx - 35n^2$. 10. $x^2 + 10x - 600$. 12. $x^2 + 3cx - 28c^2$. 14. $x^2 - 3ax - 54a^2$. 16. $x^2 + 13bx + 42b^2$. 18. $x^2 - 14cx + 45c^2$.

**Page 60.** — 1. $4a^2b$. 2. $-7a^2bcx$. 3. $-5xz$. 4. $7c^4y$. 6. $-3x^7y^2z^5$. 8. $8a^2b^3d$. 10. $2a^2b^3x$. 12. $8c^3xy$. 13. $3a(a-b)^2$. 14. $3c^2(a-x)^3$. 16. $4a^3(c-d)^6$.

**Page 61.**—1. $x+6$. 2. $x^2-2xy+3y^2$. 4. $4a^2b^3+2ab^4-b^5$. 6. $-5+3nx^2+n^2x$. 8. $xy-x^2-y^2$. 10. $x^2-2x+1$. 12. $ay^2-x^2y+a^2x$. 14. $6a-15b^5$. 15. $-x^3-4x^2y+3y^2$. 16. $-1+xy+5x^2y^2$. 18. $x-2x^3+3x^5-4x^7$. 20. $(a+x)^2-2(a+x)$. 22. $3x-2(a+2)^3$. 23. $3(x-7)^4-2(x-7)^2$.

**Page 63.**—1. $x-7$. 2. $x-2$. 3. $x-8$. 4. $x+3$. 5. $x-14$. 6. $x-a$. 7. $x+15$. 8. $x^2-2x+4$. 10. $x^2+2x+4$. 12. $x^3+x^2+x+1$. 13. $3x+4$. 14. $3x+6$. 15. $4x+7$. 16. $4x+11$. 18. $ax-6$. 20. $a^2x-5$. 22. $a^2x^2+3$.

**Page 64.**—1. $x^2+x+3$. 2. $x^2-3x-7$. 3. $3x+1$. 4. $3x-2$. 5. $x^2-2x+4$. 6. $4a^2+4a+1$. 7. $a^2-ax+x^2$. 8. $2a^2-3a+1$. 9. $a^3-2a^2-a+1$. 10. $a+b+c$. 11. $a^2-ab+b^2-ac+c^2-bc$. 12. $a^3-a^2-14a+2$. 14. $16-12a+9a^2$. 16. $8-12x+18x^2-27x^3$. 18. $4+10c+25c^2$. 20. $36x^2+6x+1$.

**Page 66.**—1. $2(a+c)$. 2. $a(a-c)$. 3. $2ac(3a-4c)$. 4. $b(a+c)$. 6. $2a^2(2a+x)$. 8. $a(3a-1)$. 10. $a(1+bc)$. 12. $c^2x^2(c+x)$. 14. $ac(a-ac+a^2c^2+c^3)$. 16. $6xy(x^2+2y-3xy)$. 18. $8m^3(3m-5-7m^3)$. 20. $ab(a^3-a^2b+ab^2-b^3)$. 22. $(x+2)(2a+4c)$. 24. $(a-x)(3c+5b)$. 26. $(n+3)(4a+6b)$. 28. $(x-4)(9c-6x)$. 30. $(a-b)(x+y-3)$.

**Page 67.**—1. $(a+b)(x-y)$. 2. $(x+2)(x^2+6)$. 4. $(x-4)(x^2+5)$. 6. $(x-3)(x^2-6)$. 8. $(x+4)(x^2-3)$. 10. $(x-5)(x^2-7)$. 12. $(x-6)(x^3+5)$. 14. $(x+3)(x+4)$. 16. $(x+5)(x+6)$. 18. $(x-2)(x-6)$. 20. $(x+3)(x+5)$. 22. $(x-7)(x-8)$.

**Page 69.**—1. $(x^2+a)(x^2-a)$. 2. $(x+8)(x-8)$. 3. $(5a+4b)(5a-4b)$. 4. $(a+c^4)(a-c^4)$. 6. $(nx^2+7y)(nx^2-7y)$. 8. $(z^3+6)(z^3-6)$. 10. $(1+m)(1-m)$. 12. $(a^2c+ac^2)(a^2c-ac^2)$. 14. $(a^2+4)(a+2)(a-2)$. 16. $(x^4+16)(x^2+4)(x+2)(x-2)$. 18. $(x-y+2z)(x-y-2z)$. 20. $(a+b+3c)(a+b-3c)$. 22. $(a-x+nx)(a-x-nx)$. 24. $(c+a+2b)(c-a-2b)$. 26. $(a+b-3c)(a-b+3c)$. 28. $(1+2a+2x)(1-2a-2x)$.

**Page 70.**—1. $(x^2-a)(x^4+ax^2+a^2)$. 2. $(x-4)(x^2+4x+16)$. 3. $(3x-8y)(9x^2+24xy+64y^2)$. 4. $(a^3-b^2)(a^6+a^3b^2+b^4)$. 6. $(8a-9b)(64a^2+72ab+81b^2)$. 8. $(m-x^3)(m^2+mx^3+x^6)$. 10. $(1-y^2)(1+y^2+y^4)$. 12. $(6-7c)(36+42c+49c^2)$.

**Page 71.**—1. $(a+2)(a^2-2a+4)$. 2. $(x+3)(x^2-3x+9)$. 3. $(4n+5x)(16n^2-20nx+25x^2)$. 4. $(x^3+1)(x^6-x^3+1)$. 6. $(3a^2+8c)(9a^4-24a^2c+64c^2)$. 8. $(x^2+m)(x^4-mx^2+m^2)$. 10. $(2+y^2)(4-2y^2+y^4)$. 12. $(7+6a)(49-42a+36a^2)$.

**Page 72.**—1. $(x+5)(x+5)$. 2. $(3x+5)(3x+5)$. 4. $(2x-6)(2x-6)$. 6. $(nx-3)(nx-3)$. 8. $(a^2+2c^2)(a^2+2c^2)$. 10. $(x^2-12)(x^2-12)$. 12. $(n^2+3m^3)(n^2+3m^3)$. 14. $(a^3-6)(a^3-6)$.

**Page 73.**—1. $(x+1)(x+2)$. 2. $(x-4)(x-6)$. 4. $(x+15)(x-3)$. 6. $(x-7)(x-9)$. 8. $(x-12)(x+10)$. 10. $(x+12)(x-9)$. 12. $(x+15)(x-20)$. 14. $(x-5)(x-15)$. 16. $(n+6)(n+12)$. 18. $(n-15)(n+12)$. 20. $(a-7)(a-13)$. 22. $(a+13)(a-11)$. 24. $(b-20)(b+16)$. 26. $(b+15)(b-10)$. 28. $(z-6)(z-13)$. 30. $(z-20)(z+12)$.

**Page 74.**—1. $(x+a)(x-a)$. 2. $(x+3)(x+3)$. 4. $(x-4)(x-4)$. 6. $(y+5a)(y+5a)$. 8. $(z-n^2y)(z-n^2y)$. 10. $(x+3)(x+7)$. 12. $(y-5)(y-7)$. 14. $(x-9)(x+7)$. 15. $(6-mx)(36+6nx+n^2x^2)$. 16. $(y+9)(y-8)$. 18. $(a^2+2y)(a^2-6y)$. 19. $(m^2+n)(m^4-m^2n+n^2)$. 20. $(x^3+2a)(x^3+7a)$. 22. $(y^4+8b)(y^4-5b)$. 23. $(a+b+x)(a+b-x)$. 24. $(z^5-6c)(z^5-6c)$. 25. $(x^3+x^2+x+3)(x^3+x^2-x-3)$. 26. $(x^3-x^2+x-1)(x^3-x^2-x+1)$. 27. $(x^2+2x-1)(x^2+1)$. 28. $(a^2+3)(a^2-4a-3)$. 29. $(x+3)(x^2-4)=(x+3)(x+2)(x-2)$. 30. $(y-a)(y^2+2)$. 32. $(3x-2)(x^2+5)$. 33. $-9(x^2+3x+3)$. 34. $(a+b-c+d)(a-b+c-d)$. 35. $(2a+2)(a^2+2a+4)$. 36. $(x^2+6x+4)(x^2+2x+4)$.

**Page 77.**—1. $a^2b^2$. 2. $5x^3y$. 4. $2ab$. 6. $x+1$. 8. $x+2$. 9. $x+2$. 10. $x-3$. 11. $x+3$. 12. $x-4$. 13. $x-5$. 15. $x+7y$. 16. $x+8y$. 18. $(a-3b)$. 20. $a+c$. 21. $x-3$. 22. $x-5$. 24. $a-3$. 26. $13(a^2-ab+b^2)$. 27. $a^2-c^2$. 28. $a+2b$.

**Page 79.**—1. $6a^3b^2c^3$. 2. $x(x-1)(x+1)$. 4. $a(a-1)(a+1)$. 6. $(a-1)(a+1)^2$. 8. $(a^3-8)(a+2)$. 10. $x(x^3+8)$.

**Page 80.**—1. $(x+3)(x-5)(x+7)$. 2. $(x-4)(x+7)(x-10)$. 4. $(a-2)(a+8)(a-11)$. 6. $(x+4)^2(x+2)$. 7. $(a+2y)(a-3y)(2a+y)$. 8. $2(x+11y)(x-9y)(x+6y)$. 9. $3(x+5y)(x+9y)(x-12y)$. 10. $5(x+4y)^2(x-5y)$. 11. $4(a+5b)(a+8b)(a-10b)$. 12. $(x+3y)(x+5y)(x-7y)$. 13. $a(a^2-b^2)(a+2b)$. 14. $21(x+4)^2(x-4)(x-6)$. 15. $4a(x^2-9)(x+7)(x+10)$. 16. $b(x^2-b^2)(x-4b)(x-10b)$. 18. $(a+3)^2(a-3)(a-7)$. 19. $(a+b+c)(a-b)(b-c)$. 20. $(a+x)(x+4)(a-x+4)$. 21. $3(x+1)(x+2)^2(x-5)$. 22. $(x-3)(x+4)(x-6)(x^2+2)$. 23. $(x+3)(x+5)(x+6)(x^2+3)$.

**Page 82.**—1. $\frac{3}{4}$. 2. $\frac{3a}{4b}$. 3. $\frac{3mn^2rx^3}{4y}$. 4. $\frac{5}{11}$. 5. $\frac{2x^4}{3y^2}$. 7. $\frac{3a-4b}{a^2-2a}$. 8. $\frac{x+10}{x+9}$. 9. $\frac{3x+2y}{2-x^2}$. 10. $\frac{a-12}{a-10}$. 11. $\frac{3x-2y}{3x}$. 12. $\frac{x+y-2}{a}$. 13. $\frac{4x}{x+3}$. 14. $\frac{a-b+c}{n}$. 15. $\frac{a}{a-6}$. 16. $\frac{x+3}{x-5}$.

**Page 83.**—1. 5. 2. $x+3$. 3. $x-1$. 4. $6\frac{1}{7}$. 5. $x+1-\frac{1}{x+1}$. 6. $x^2-2x+4$. 7. $x+5$. 8. $x-3$. 9. $x-15$. 10. $a^2+ab+b^2$. 11. $x+7-\frac{23}{x-11}$. 12. $x+1$. 13. $x+1$. 14. $3+\frac{5x+2}{x^2+2}$. 15. $12x-36+\frac{108}{x+3}$. 16. $4a^2-10a+25$.

**Page 84.**—1. $\frac{17}{4}$. 2. $\frac{11x}{2}$. 3. $\frac{14x-5}{4}$. 4. $\frac{32}{5}$. 5. $\frac{37x}{7}$. 6. $\frac{32x-4}{7}$. 9. $\frac{7cx-3a+b}{x}$. 10. $\frac{3x^2+5x-3}{x+1}$. 11. $\frac{2a^2}{a-x}$. 12. $\frac{10a^2+11a-7}{2a+3}$. 13. $\frac{a^2}{a+c}$.

**Page 86.**—1. $\frac{15}{24}, \frac{14}{24}$. 2. $\frac{bm}{a^2b^2}, \frac{an}{a^2b^2}$. 4. $\frac{5c}{abc}, \frac{6b}{abc}, \frac{7a}{abc}$. 6. $\frac{a^2x^2}{axy}, \frac{bcy}{axy}, \frac{bdx}{axy}$. 8. $\frac{a^2+2ab+b^2}{a^2-b^2}, \frac{a^2-2ab+b^2}{a^2-b^2}$. 10. $\frac{4x^2-8x}{x^3-4x}, \frac{3x^2+6x}{x^3-4x}, \frac{x^2-4}{x^3-4x}$. 12. $\frac{2x}{x^3-1}, \frac{5x-5}{x^3-1}$. 13. $\frac{2x+4}{2x^2-18}, \frac{3x-9}{2x^2-18}, \frac{5x+15}{2x^2-18}$. 14. $\frac{x^2+5x+4}{x^3+3x^2-10x-24}, \frac{x^2+x-2}{x^3+3x^2-10x-24}, \frac{2x^2-2x-12}{x^3+3x^2-10x-24}$. 15. $\frac{x(x+3)}{(x+1)(x+2)(x+3)}, \frac{x(x+1)}{(x+1)(x+2)(x+3)}, \frac{3(x+1)(x+2)}{(x+1)(x+2)(x+3)}$. 16. $\frac{4x^2-8x}{12x^2-48}, \frac{9x^2+18x}{12x^2-48}, \frac{2.}{12x^2-48}$. 17. $\frac{3}{x^4-y^4}, \frac{2x^2+2y^2}{x^4-y^4}, \frac{x^3-x^2y+xy^2-y^3}{x^4-y^4}$. 18. $\frac{a^2}{a^4+ab^3}, \frac{a^3-ab^2}{a^4+ab^3}, \frac{a^2-ab+b^2}{a^4+ab^3}$. 19. $\frac{x^2-36}{x^3+12x^2+44x+48}, \frac{x^2-16}{x^3+12x^2+44x+48}, \frac{x^2-4}{x^3+12x^2+44x+48}$.

**Page 87.**—1. $\frac{29}{20}$. 3. $\frac{7x-1}{12}$. 4. $\frac{11x}{40}$. 6. $\frac{2x+31}{15}$. 9. $\frac{a-12}{24}$. 11. $\frac{11}{24x}$. 12. $\frac{5a}{24x}$.

**Page 88.**—1. $\frac{7x+1}{12}$. 2. $\frac{7x-113}{40}$. 3. $\frac{3x+7}{6}$. 4. $\frac{5x-y}{12x}$. 5. $\frac{3x+11y}{30}$. 7. $\frac{a^3+b^3-ac^2}{abc}$. 8. $\frac{2x^2+6x+6}{(x+1)(x+2)(x+3)}$. 9. $\frac{3x^2+26x+59}{(x+3)(x+4)(x+5)}$. 10. $\frac{8-4x}{9-x^2}$. 11. $\frac{x^2+9}{x^2-9}$. 12. $\frac{2x+7}{(x+2)(x+3)(x+5)}$. 13. $-\frac{x+2}{(x^2-1)(x-2)}$. 14. $\frac{175x+504}{(x^2-9)(x^2-16)}$. 15. 0. 16. $\frac{x-2}{x^3-1}$.

**Page 90.** — 1. $\frac{7}{3}$. 2. $3a$. 3. $\frac{m}{a^2b}$. 6. $\frac{anx^2}{6}$. 8. $\frac{n}{ax}$. 9. $\frac{1}{10bc}$.
11. $\frac{x-1}{x+2}$. 12. $4b^2+6$. 13. $x^2+5x+6$. 15. $\frac{1}{x^2+6x+9}$.
16. $74a^2$. 17. $\frac{1}{x^2-9}$. 19. $11x$. 21. $15a$. 23. $\frac{x^2+5x+4}{x+7}$.
24. $\frac{x-8}{2x+6}$. 25. $\frac{x-5}{x^2-4x}$.

**Page 91.** — 2. $\frac{3a^2x}{7c}$. 3. $\frac{x+2}{x-3}$. 5. $\frac{2xy}{3c^2}$. 6. $\frac{x-1}{3x}$. 8. $\frac{3c}{7a}$.
9. $\frac{x^2+x+1}{x^2-x+1}$.

**Page 92.** — 1. 6. 2. $\frac{a^2-1}{a^2+1}$. 3. 14. 4. $\frac{6a+12}{a^2}$. 6. $\frac{1}{abc}$.
7. $\frac{x+8}{x-4}$. 8. $\frac{x+2}{x-7}$. 9. $\frac{x^2-2x+4}{x^2+2x+4}$. 10. $\frac{a-1}{a+1}$. 11. $\frac{x-2}{x-3}$.
12. $\frac{x^2-9x+20}{x^2-64}$. 13. $\frac{x+1}{x+5}$. 14. 1. 15. 1. 16. $\frac{x+y}{a-b}$.
17. $\frac{a^3+2a^2x+2ax^2+x^3}{x}$. 18. $\frac{1}{x}$. 19. $\frac{1}{2-c}$.

**Page 94.** — 1. $\frac{6x+4y}{4x-3y}$. 2. $\frac{bcdx+acdy}{abdx-abcy}$. 3. $\frac{21x}{22y}$. 4. $\frac{12a+9b}{12a-8b}$.
5. $\frac{b+a}{b-a}$. 6. $\frac{2c+3b}{2c-3b}$. 7. $\frac{x^2+x-2}{x^2-x-2}$. 8. $\frac{a^2+a-2}{a^2-a-2}$. 9. $\frac{a+b}{a-b}$.
10. $\frac{x^2-8x+15}{x^2-5x+4}$. 11. $\frac{x+5}{x+9}$. 12. $\frac{x^2+1}{2x}$. 13. $\frac{x+8}{x-6}$. 14. $x+y+4$.
15. $\frac{x+5}{x+8}$. 16. $c-1$.

**Page 95.** — 1. $\$\left(\frac{5a}{6}-2\right)$. 2. $\frac{n-3}{n}$. 3. $\$\left(\frac{15-4n}{20}\right)$. 4. $\$4n$.
5. $\$\frac{8n}{15}$. 6. $\$\frac{4a-2b}{3}$. 7. $6an$. 8. $\left(\frac{5a-b}{6}\right)$ mi. 9. $(3a+2b)$ mi.
10. $\$\frac{7a-35}{2}$. 11. $\frac{1}{a}-\frac{1}{b}+\frac{1}{c}$. 12. $\frac{6x^2-8x+15}{12x}$.

**Page 96.** — 13. $\left(\frac{2x}{3}-4\right)\not\$$. 14. $\frac{x}{2}$. 15. $\frac{2x-5}{3}$. 16. $\frac{4x}{25+5x}$.
17. $\frac{15}{x^2+7x+15}$. 18. A, $\frac{2x+y}{3}$, B, $\frac{x+2y}{3}$. 19. $\$\frac{5x-100}{6}$.
20. $\$\frac{5x-340}{3}$. 21. $\frac{3x-75}{5x-75}$. 22. $\$\frac{nx}{m}$.

**Page 97.** — **23.** $\$(n^2+n-2)$. **24.** $\$(nd+2d)$. **25.** $\$(405-1\frac{1}{4}n)$. **26.** $\frac{n+3}{n}$. **27.** $x-4$. **28.** $n+5$. **29.** $\frac{x-7}{x}$. **30.** $15-\frac{11n}{12}$. **31.** $\frac{11x}{12}-30$.

**Page 98.** — **33.** $\frac{1}{3}$, 3 da. **34.** (*a*) $\frac{n+x}{nx}$, $\frac{2(n+x)}{nx}$, $\frac{c(n+x)}{nx}$; (*b*) $\frac{nx}{c+x}$ da. **35.** (*a*) $\frac{1}{6}$, $\frac{1}{3}$, $\frac{2}{3}$; (*b*) 6 da. **36.** (*a*) $\frac{13}{8x}$, (*b*) $\frac{24x}{13}$. **37.** $\frac{3}{4x}$. **38.** $\frac{10-x}{10x}$.

**Page 100.** — **2.** 15. **4.** 40. **6.** 4. **8.** 18. **10.** 12. **12.** 16. **14.** 5. **16.** 3. **18.** 7. **20.** 2. **22.** 16. **24.** 2. **26.** 3.

**Page 101.** — **2.** $\frac{4}{5}$. **4.** 11. **6.** 4. **8.** 5. **9.** 2. **10.** 4. **11.** 1. **12.** 5. **13.** 2. **14.** 2. **15.** $\frac{2}{5}$. **16.** $-\frac{11}{10}$. **17.** 11. **18.** 5. **19.** 5.

**Page 102.** — **2.** 156. **3.** \$24. **4.** 105. **5.** 720 ft. **6.** 36 yr., 9 yr. **7.** 24. **8.** \$30. **9.** \$4.80.

**Page 103.** — **11.** 25, 35. **12.** 15 yr. **13.** 12. **14.** 16. **15.** \$162. **16.** 90 rd., 27 rd.

**Page 104.** — **18.** A's, 10 yr., B's, 40 yr. **19.** 6. **20.** 28 yr. **21.** 40, 20. **22.** \$6. **23.** 15 g., 35 c. **24.** 30, 40.

**Page 105.** — **1.** $x=12$, $y=6$. **2.** $x=3$, $y=6$. **3.** $x=9$, $y=25$. **4.** $x=12$, $y=8$. **5.** $x=6$, $y=12$. **6.** $x=7$, $y=9$. **7.** $x=6$, $y=12$. **8.** $x=24$, $y=12$. **9.** $x=21$, $y=20$. **10.** $x=8$, $y=2$. **11.** $x=2$, $y=1$.

**Page 106.** — **1.** $\frac{6}{5}$. **2.** $\frac{6}{5}$, $\frac{5}{30}$. **3.** $\frac{2}{5}$. **4.** 8, 43. **5.** 24 yr., 48 yr. **6.** \$100. **7.** A, 4 da., B, 12 da. **8.** A, 8 da., B, 24 da.

**Page 107.** — **9.** 1.1 and .95. **10.** 63, 37. **11.** 28, 27. **12.** 170, 225. **13.** A, 5 da., B, 20 da. **14.** 25, 75. **15.** 80 bu. c, 60 bu. o.

**Page 108.** — **2.** 5 c. **3.** $a+5$. **6.** $\frac{a-c}{c}$. **7.** $n-1$. **8.** $1-4a$. **10.** $c+2$. **11.** $a^2-ab+b^2$. **12.** $a^2+2a+7$. **13.** $a^2+ab+b^2$. **14.** $\frac{4m}{a-b+2}$. **15.** $\frac{4c+3d}{4b-6a}$. **16.** $\frac{2-c}{a}$. **17.** $\frac{s}{bc-ac+ab}$. **18.** $\frac{4c-b}{40-6a}$. **19.** $a^2+3$. **20.** $a^2+3a+2$. **21.** $3c^2-4c+1$. **23.** $\frac{a}{p+pt}$. **25.** $\frac{a-b-c}{a-c}$. **26.** $\frac{3n-2a}{c}$. **27.** $\frac{ab-3a+2b}{a+b}$.

**Page 109.** — 1. $\frac{12-5y}{3}$. 2. $\frac{25-9x}{7}$. 3. $\frac{4m-10}{3}$. 4. $\frac{7a-16}{2}$. 6. $\frac{16x-20b}{15}$. 7. $\frac{2m+n-4}{2}$. 8. $-\frac{2+4a}{3}$. 9. $14a-60$. 10. $\frac{3-2n^2}{2}$. 11. $\frac{1}{2}$.

**Page 110.** — 1. $x=1, y=2$. 2. $x=8, y=7$. 3. $x=12, y=4$. 4. $x=5, y=2$. 5. $x=5, y=3$. 6. $x=3, y=10$. 7. $x=1, y=2$. 8. $x=3, y=6$. 9. $x=4, y=-6$. 10. $x=4, y=2$. 11. $x=\frac{3c-bd}{6-ab}, y=\frac{2d-ac}{6-ab}$. 12. $x=\frac{a+b}{a}, y=\frac{a-b}{b}$.

**Page 111.** — 1. 27 mi. 2. 15. 3. 42 mi.

**Page 112.** — 2. 6. 3. 6 mi. 4. 150 mi. 5. $44\frac{4}{9}$.

**Page 113.** — 2. $6:32\frac{4}{11}$. 3. $4:54\frac{6}{11}$. 4. $5:10\frac{10}{11}$. 5. $5:49\frac{1}{11}$. 6. $9:16\frac{4}{11}$. 7. (*a*) $10:5\frac{5}{11}$; (*b*) $10:38\frac{2}{11}$.

**Page 114.** — 1. 30, 13. 2. 121, 94. 3. A, 14, 839; B, 15, 436. 4. $d=2x-s$.

**Page 115.** — 1. $9x^2y^2$. 2. $+16x^8y^4$. 3. $-128a^{21}b^{14}$. 6. $729c^{30}x^{18}$. 9. $x^{10}y^{20}z^{30}$. 11. $\frac{8a^9b^3}{125x^6y^9}$. 13. $\frac{a^{32}b^{16}x^8}{256c^8}$.

**Page 116.** — 1. $5a$. 2. $7a^2bx^3$. 3. $11a^3b^2x^6$. 6. $10c^7d^3y$. 9. $13x^8y^5z^2$. 11. $2(a-x)^2(x+y)^3$. 13. $a(n+a)^4(x-y)$. 15. $3(x+a)^2(x-a)^5$.

**Page 117.** — 1. $5x+2y$. 2. $3a-4b$. 3. $6a-2b$. 4. $ax-2by$.

**Page 118.** — 1. $x-3y$. 2. $ax-5c$. 4. $4x+5a^2$. 6. $6b^2-7n$. 8. $3+12c^2n$. 10. $13ax^2-1$.

**Page 119.** — 1. $2x^2+x+1$. 2. $n^2+3n+1$. 3. $2y^2-4y+1$. 4. $4a^2-2a-2$. 5. $3x^2+x-2$. 6. $3x^2-4x+2$. 7. $n^3+2n^2+1$. 8. $3a^4+5a^2x-4x^2$. 10. $2x^3-x^2+3x-4$. 11. $8a^3-4a^2+2a-1$. 12. $4x^3-3x^2+2x-1$. 14. $\frac{x^2+x+1}{x^2+2x+1}$. 15. $\frac{x^2-y+z^2}{x+y-1}$.

**Page 120.** — 1. 2, two figures. 3. 5, two. 6. 9, three. 9. 7, four.

**Page 121.** — 1. 16. 2. 32. 3. 54. 6. 85. 9. 96. 11. 81. 12. 73. 14. 54. 15. 95.

**Page 122.** — 1. 125. 2. 723. 3. 2401. 4. 243. 5. 545. 6. .027. 7. 216. 8. 348. 9. 13.04. 10. 142. 11. 625. 12. 2780. 14. 733. 15. .0145. 18. .0678. 21. 6.07. 23. 817. 24. .0047. 25. 1.4142+. 26. .9354+. 27. 37.97+. 28. $291\frac{2}{3}$. 29. 1.7320+. 30. .8944+. 31. 50.006+. 32. 70.001+.

**Page 124.** —1. $-2, +2$. 2. $8, -3$. 4. $5, 5$. 6. $4, -9$. 8. $-5, -7$. 10. $8, -5$. 12. $a, -c$. 14. $-a, -b$. 16. $n, -r$. 18. $d, e$. 19. $\frac{5}{2}, -\frac{5}{2}$. 20. $-\frac{1}{4}, \frac{3}{4}$. 22. $\frac{1}{2}, -\frac{1}{2}$. 24. $-\frac{4}{3}, \frac{3}{4}$. 26. $\frac{3}{2}, \frac{5}{6}$. 27. 1, 2, 3. 29. 3, −3, 6. 31. 2, $-\frac{1}{3}, \frac{3}{2}$. 33. $\frac{4}{3}, -4, -2, \frac{8}{7}$.

**Page 125.** —1. $4, -4$. 2. $5, -7$. 4. $3, 5$. 6. $3, 6$. 8. $9, -5$. 10. $5, -12$. 11. $\frac{5a}{2}, -\frac{5a}{2}$. 12. $10a, -7a$. 14. $8b, -12b$. 16. $2c, 6c$. 18. $4d, 5d$; $9m, -8m$. 22. $1, -1, 3$. 23. $2, -2, -5$. 24. $2, -2, 3\frac{1}{2}$.

**Page 127.** —1. $\pm 4$. 2. $\pm 3$. 4. $\pm 1$. 6. $\pm 3$. 8. $\pm 9$. 9. $\pm 5$. 10. $\pm 10$. 11. $\pm(n-3)$. 12. $\pm(c+a)$. 13. 36. 14. 10. 15. 9, 12. 16. 10, 15. 17. 10, 24.

**Page 128.** —1. $x^2+4x+4$. 2. $x^2+5x+\frac{25}{4}$. 3. $x^2-2mx+m^2$. 5. $x^2+9x+\frac{81}{4}$. 6. $x^2+3ax+\frac{9a^2}{4}$. 7. $x^2+x+\frac{1}{4}$. 8. $x^2-11x+\frac{121}{4}$. 9. $x^2-6ax+9a^2$. 10. $x^2-x+\frac{1}{4}$. 12. $x^2+9nx+\frac{81n^2}{4}$. 13. $x^2-\frac{2}{3}x+\frac{1}{9}$. 14. $x^2+\frac{3}{4}x+\frac{9}{64}$. 15. $x^2+\frac{2a}{5}x+\frac{a^2}{25}$. 16. $x^2-\frac{6}{5}x+\frac{9}{25}$. 17. $x^2-\frac{5}{4}x+\frac{25}{16}$. 18. $x^2+\frac{2n}{m}x+\frac{n^2}{m^2}$. 19. $x-3$. 20. $x+\frac{9}{2}$. 21. $x-c$. 23. $x-\frac{7}{2}$. 24. $x+\frac{3a}{2}$. 26. $x+\frac{3}{16}$. 27. $x+\frac{b}{c}$. 29. $x-\frac{5}{18}$. 30. $x-\frac{m}{2n}$.

**Page 130.** —1. $2, 4$. 2. $2, -7$. 3. $-1, -7$. 4. $9, -6$. 6. $8, -9$. 8. $3, 4$. 10. $-4, -5$. 12. $7, 8$. 14. $-5, -20$. 16. $21, -6$. 18. $8, -35$. 20. $50, -7$. 22. $1, -\frac{1}{3}$. 23. $3, -\frac{3}{4}$. 24. $1, -\frac{5}{3}$. 25. $2, -\frac{17}{6}$. 26. $2, -\frac{6}{7}$. 27. $\frac{3}{2}, -\frac{5}{2}$. 28. $-1, \frac{17}{9}$. 29. $4, -\frac{13}{4}$. 30. $a+b, a-b$. 31. $7m, -5m$. 32. $2a+b, -b$. 33. $7a, -10a$. 34. $\frac{a+b}{2}, \frac{a-b}{2}$. 35. $\frac{3c}{5}, -\frac{c}{2}$. 36. $3a+c, a-3c$.

**Page 131.** —1. $4, -\frac{1}{2}$. 2. $\frac{2}{3}, -\frac{4}{3}$. 3. $-2, \frac{4}{3}$. 4. $1, \frac{1}{2}$. 5. $6, 9$. 6. $-2, -4$. 7. $5, \frac{10}{3}$. 8. $3, 12$. 9. $32, -2$. 10. $4, -14$. 11. $4, -10$. 13. $2, -6$. 15. $3, -3\frac{1}{2}$. 17. $12, -2$. 18. $1, -\frac{1}{3}$. 19. $1, 4$.

**Page 132.** —2. $7, 8$. 3. $8, 9$. 4. $9, -6$. 5. $8, -13$. 6. $3, 4$. 7. $6, 9$. 8. $8, 15$, or $-18\frac{1}{2}, -11\frac{1}{2}$.

**Page 133.** —10. 16. 11. 30 mi. 12. 6. 13. 12. 14. 28. 15. 8 ft.

**Page 134.** —1. 228. 2. $l=\frac{a}{b}$. 3. 120 rd. 4. 5 ch., 8 ch. 5. 4 in., 6 in. 6. $\frac{bh}{2}$. 7. 160 yd. 8. 10 in. 9. 3 ft.

**Page 148.** — 1. $x > 6$. 2. $x < -3$. 3. $x < 6$. 4. $x < 5$. 5. $x > 1$. 7. $-1 < x < 2$. 8. $15 < x < 20$. 9. $5 < x < 12$. 10. $3 < x < 5$. 11. $x < 8$, $y > 3\frac{1}{2}$. 12. $x < 20$, $y > 7$.

**Page 150.** — 1. 1516. 2. 12%. 3. \$1500. 4. \$120. 5. \$96. 6. $33\frac{1}{3}$%.

**Page 151.** — 1. \$185.50. 2. 5%. 3. 2 yr. 3 mo. 18 da. 4. \$84. 5. \$672.24.

**Page 152.** — 1. 122°. 2. $-40°$. 3. $c = \frac{1}{9}(5f - 160)$. 4. 15%. 5. $-20°$. 6. $-17\frac{7}{9}°$. 7. 320°. 8. 37°.

**Page 153.** — 2. $(x+4)(x+4)$. 3. $(x-2y^2)(x^2+2xy^2+4y^4)$. 5. $(x+2y^2)(x^2-2xy+4y^4)$. 7. $(x^2+9)(x+3)(x-3)$. 8. $(x-a)(x-a)$. 11. $(2x+3)(4x^2-6x+9)$. 14. $(x+8)(x-5)$. 15. $(4x^2+25)(2x+5)(2x-5)$. 17. $(c-n)(x+y)$. 18. $(x^2+3)(x+4)$. 20. $(x^2-2)(3x-5)$. 22. $(x+2a)(2x-3y)$. 23. $(x^2+2x+1)(x^2-2x+1)$. 24. $(x^2+4x+8)(x^2-4x+8)$. 26. $(a^2+5a-5)(a^2-5a-5)$. 28. $(x+y)(x+y)(x-y)(x-y)$. 29. $(x^2+5x+3)(x^2+x+1)$. 30. $(y^2-y+3)(y^2-7y+7)$.

**Page 154.** — 1. $(a^2+a+1)(a^2-a+1)$. 2. $(3a^2+3ac+5c^2)(3a^2-3ac+5c^2)$. 4. $(4x^2+3xy-7y^2)(4x^2-3xy-7y^2)$. 6. $(5x^2+xy-4y^2)(5x^2-xy-4y^2)$. 8. $(12+x-9x^2)(12-x-9x^2)$. 10. $(7a^2+13ab+11b^2)(7a^2-13ab+11b^2)$.

**Page 155.** — 1. $(3x+1)(2x+3)$. 2. $(4x+5)(3x-8)$. 4. $(4x+5)(3x-8)$. 6. $(7x+3)(8x-5)$. 8. $(6x-5)(4x+1)$. 10. $(8x+3)(3x-1)$. 12. $(12x-5)(2x-1)$. 14. $(24x+25)(x-1)$. 16. $(3a^2+b^2)(2a+b)(2a-b)$.

**Page 156.** — 1. 0, −1, 0. 2. 0. 3. 0. 4. 0. 5. 0. 6. 44. 7. 0. 8. −114. 9. 0.

**Page 158.** — 1. $(x-1)(x-2)(x+3)$. 2. $(x-1)(x-2)(x-2)$. 3. $(x+1)(x^2-x+6)$. 4. $(x+1)(x+2)(x+11)$. 6. $(x-1)(x-3)(x-7)$. 8. $(x-4)(x-3)(x+5)$. 10. $(x-3)(x-3)(x-4)$. 12. $(x-1)(3x^2+x+6)$. 14. $(x+1)(2x^2+x-4)$. 16. $(3c-1)(3c^2+8c+1)$. 18. $(s+1)(s+1)(s+2)(s-4)$. 20. $(n-1)(n+1)(n+2)(n-3)$.

**Page 159.** — 1. Put $a$ for $x$, $a^n - a^n = 0$. 2. $(-a)^n - a^n = 0$, if $n$ is even. 3. $(-a)^n + a^n = 0$ if $n$ is odd. 5. $(x-a)(x^4+ax^3+a^2x^2+a^3x+a^4)$. 6. $(c+2)(c^4-2c^3+4c^2-8c+16)$. 8. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.

1. $x+4$. 2. $x-5$. 3. $x+9$. 4. $x+3$. 5. $x+8$. 6. $x+7$. 8. $x-2$. 9. $a-3$.

**Page 163.** — 1. $x+7$. 2. $x+11$. 3. $3(x-1)$. 4. $2x+5$. 5. $4x-1$. 6. $a+b$. 7. $3x^2+12x$. 8. $a+x+y$. 9. $y+z$. 10. $a-3b$. 11. $x-2$. 12. $x-3$. 13. $2x-1$. 14. $2x^2+7x+3$. 15. $(x+1)^3$.

**Page 164.** —1. $(x+1)(x+2)(x+3)(x+4)$. 2. $(x+2)(x-2)(x+4)(x-4)$. 3. $(x+2)(x-2)(x+5)(x-5)$. 4. $(x-1)^2(x+1)(3x+2)(2x-3)$. 6. $(x+3)(x-3)(x+5)(x+7)$. 8. $(y+1)(y-1)(y-2)(y+3)(y+4)$. 10. $3c(c+5)(3c-1)(2c+3)$. 12. $(y+1)(y-1)(y^2+y+3)(y^3+y^2+y-2)$.

**Page 165.** —1. $\frac{x+9}{x+6}$. 2. $\frac{2x+5y}{2x+3y}$. 4. $\frac{x^2+x+1}{x-4}$. 6. $\frac{a^2-2a+2}{a^2-2}$. 8. $\frac{3-3}{3-2}$. 10. $\frac{2y+5}{3y-1}$. 11. $a^2-ab+b^2$. 12. $\frac{9x^3+3x^2-2x+1}{3x^2+x-1}$.

1. $x+2$. 2. $2x+6y+\frac{65y^2}{3x-10y}$. 4. $a^2+2ab+b^2$. 5. $7a-47+\frac{289}{a+6}$. 6. $1-\frac{12}{x+4}$. 8. $x+\frac{4x}{x+5}$. 10. $6x-13+\frac{10}{x+1}$.

**Page 166.** —1. $\frac{2x^2+4x+1}{x}$. 2. $\frac{2x^2+8}{x+2}$. 3. $\frac{2x^2-2x-1}{x}$. 4. $\frac{12x}{x+3}$. 6. $\frac{a}{a-1}$.

1. $\frac{x^3}{x^3-9x}$, $\frac{x^2-x-6}{x^3-9x}$, $\frac{x^2+6x+9}{x^3-9x}$. 2. $\frac{9+18x+9x^2}{8(1+x)^3}$, $\frac{6x+6x^2}{8(1+x)^3}$, $\frac{4x^2}{8(1+x)^3}$. 3. $\frac{x-4}{x^3-9x^2+26x-24}$, $\frac{x-2}{x^3-9x^2+26x-24}$, $\frac{x-3}{x^3-9x^2+26x-24}$. 4. $\frac{x^2-4x-5}{x^3-6x^2-x+30}$, $\frac{x^2-4x+3}{x^3-6x^2-x+30}$, $\frac{x^2+2x}{x^3-6x^2-x+30}$. 5. $\frac{(a-b)(a^3+b^3)}{a^6-b^6}$, $\frac{(a+b)(a^3-b^3)}{a^6-b^6}$, $\frac{1}{a^6-b^6}$.

**Page 167.** —1. $\frac{18x}{9x^2-1}$. 2. $\frac{4x^2+6x+3}{(x+1)^3}$. 3. $\frac{2x^2+7x+1}{x^3+12x^2+47x+60}$. 4. $-\frac{11x+64}{(x+a)(x-5)(x+6)}$. 5. $\frac{x+6}{x^2+4x+3}$. 6. $\frac{x+7}{x^2+6x+8}$. 7. $\frac{-2}{(x-4)(x-5)(x-6)}$.

**Page 168.** —8. 0. 9. $\frac{2x-3}{(x^2-1)(2x+3)}$. 10. $\frac{ax-b^2}{x^2-b^2}$. 12. $\frac{2ab^2}{a^4-b^4}$ 14. 0. 15. —1. 16. 0. 17. 0. 18. 0.

**Page 169.** —1. $\frac{x+3}{x-7}$. 2. $\frac{x^2+x-2}{x^2-3x-4}$. 3. 1. 4. 1. 5. $\frac{3x}{4y}$. 7. $b$. 8. $\frac{x^4}{a^4}+\frac{x^2}{a^2}+1$. 9. $x^2+1+\frac{1}{x^2}$. 10. 1.

**Page 170.**—1. $\frac{x^2-7x+6}{x^2}$. 2. $\frac{a}{b}-\frac{b}{a}$. 3. $\frac{x^3-y^3}{x^3+y^3}$. 5. $x^2+1+\frac{1}{x^2}$. 7. $x+1+\frac{1}{x}$. 8. $a^2-b^2+c^2+2ac$. 10. $\frac{x^2-12x+35}{x^2-14x+48}$. 11. $a$. 12. $\frac{x-a+b}{x+a-b}$. 13. $\frac{(x-b)^2-(a-c)^2}{(x-a)^2-(b-c)^2}$.

**Page 171.**—1. $\frac{11}{10}$. 2. $\frac{x-1}{x+1}$. 5. $\frac{a^2+x^2}{2ax}$. 6. $\frac{x^2+1}{2x}$. 7. 1. 8. $\frac{x^2-64}{x^2-144}$. 9. $\frac{xy}{x^2+y^2}$. 10. $\frac{(a^2+b^2)^2}{2a^2b^2}$. 11. $\frac{x^2+1}{x^3+2x}$. 12. $\frac{bc+1}{abc+a+c}$. 13. $\frac{adf+ae}{bdf+be+cf}$. 14. $\frac{4}{3(x+1)}$.

**Page 172.**—1. $12\frac{1}{2}$ yr. 2. $184\frac{2}{11}$ lb.

**Page 173.**—5. 78%. 6. 95%. 7. 90%. 8. 32 yr., 38 yr., 47 yr.

**Page 174.**—9. 10. 10. 7. 11. 6. 12. 84 gal. 13. 50 lb. 14. 30 lb. and 70 lb.

**Page 175.**—1. $x=5, y=4$. 2. $x=5, y=7$. 3. $x=60, y=36$. 4. $x=12, y=20$. 5. $x=16, y=7$.

**Page 176.**—1. $x=3, y=4$. 2. $x=3, y=5$. 3. $x=144, y=216$. 5. $x=\frac{1}{2}(s+d), y=\frac{1}{2}(s-d)$. 8. $x=\frac{qr-ns}{mq-np}, y=\frac{pr-ms}{mq-np}$. 10. $x=\frac{2}{a+b}, y=\frac{2}{a-b}$. 12. $x=(a+b)^2, y=(a-b)^2$.

**Page 178.**—1. $x=3, y=2, z=5$. 2. $x=10, y=2, z=3$. 3. $x=2, y=3, z=4$. 4. $x=3, y=4, z=6$. 5. $x=2, y=3, z=4$. 6. $x=5, y=4, z=3$. 7. $x=1, y=2, z=3$. 9. $x=4, y=9, z=16, u=25$. 11. $x=\frac{b^2+c^2-a^2}{2bc}, y=\frac{a^2+c^2-b^2}{2ac}, z=\frac{a^2+b^2-c^2}{2ab}$. 12. $x=\frac{n(n-b)(n-c)}{a(a-b)(a-c)}$. 13. $x=\frac{a}{2}, y=\frac{b}{2}, z=\frac{c}{2}$. 14. $x=b+c-a$.

**Page 179.**—1. 12, 15. 2. 1, 3, 5. 3. 4, 5, 6. 4. \$1.50, \$.25, \$.15. 6. w. 80¢, c. 40¢, r. 60¢. 7. 25, 9, 6.

**Page 180.**—9. 8, 12, 24 da. 10. 18, 24, 36 da. 11. A in $\frac{2abc}{ac+bc-ab}$ min., B in $\frac{2abc}{ab+bc-ac}$ min., C in $\frac{2abc}{ab+ac-bc}$ min.

**Page 181.**—13. 245. 14. 891. 15. 4 and 5 yd. 16. \$8250, 4%.

**Page 182.**—18. 5 mi. per hr. 19. Dist. 24 mi., rate 5 mi. per hr. 20. $\frac{bc}{c+d}, \frac{bd}{c+d}, \frac{a(c^2-d^2)}{2bcd}$. 21. 36 mi., 18 mi.

**Page 184.**—1. $x = 5$, $y = 2$. 2. $x = 5, 12, \cdots, y = 1, 5, \cdots$. 3. $x = 10$, $y = 1$. 4. $x = 5, 10, \cdots, y = 2, 8, \cdots$. 5. $x = 4$, $y = 5$. 6. $x = 10, 19, \cdots, y = 3, 16, \cdots$. 7. $x = 15$ or 2, $y = 1$ or 2. 9. no values.

**Page 186.**—1. 1 *l* and 12 *p*, or 3 *l* and 7 *p*, or 5 *l* and 2 *p*. 2. 41 *s* and 1 *c*, or 22 *s* and 7 *c*, or 3 *s* and 13 *c*. 3. 8 ($2) and 1 ($3), or 5 ($2) and 3 ($3), or 2 ($2) and 5 ($3). 4. 6, 56, 8; 12, 42, 16; 18, 28, 24; 24, 14, 32. 5. 23. 6. $\frac{1}{10}, \frac{2}{11}, \cdots$. 7. 58. 8. 4, 11, 25, or 8, 2, 30. 9. 4 s., 6 p., 90 h.

**Page 190.**—3. $729\,a^3$. 6. $3\,a^4$. 8. $3^6 = 729$. 9. $16\,a^6b^4c^2$. 10. $-a^6$. 11. 256. 12. $-125\,x\,y^{12}$. 15. $\frac{a^6c^{12}x^{16}}{b^{20}y^{24}}$. 17. $-2\,a^2b$. 18. $-ab^2x^3$. 19. $5184\,x^6y^{10}$. 20. $729\,a^6c^9$. 21. $25\,ab^2$. 22. $3\,a^2b$.

**Page 192.**—2. $a^4+4\,a^3x+6\,a^2x^2+4\,ax^3+x^4$. 4. $a^3-3\,a^2x+3\,ax^2-x^3$. 6. $x^5-5\,x^4y+10\,x^3y^2-10\,x^2y^3+5\,xy^4-y^5$. 8. $x^4+4\,x^3+6\,x^2+4\,x+1$. 10. $x^3-3\,x^2+3\,x-1$. 12. $y^5-5\,y^4+10\,y^3-10\,y^2+5\,y-1$. 14. $x^6-6\,ax^4+12\,a^2x^2-8\,a^3$. 15. $a^6+3\,a^5+6\,a^4+7\,a^3+6\,a^2+3\,a+1$. 17. $a^8-12\,a^7+54\,a^6-108\,a^5+81\,a^4$. 18. $x^4-2\,x^3-x^2+2\,x+1$. 20. $a^9+12\,a^6c+48\,a^3c^2+64\,c^3$. 21. $1-8\,a+28\,a^2-56\,a^3+70\,a^4-56\,a^5+28\,a^6-8\,a^7+a^8$. 23. $16\,a^8-96\,a^7y+216\,a^4y^2-216\,a^2y^3+81\,y^4$. 24. $125-150\,x-15\,x^2+52\,x^3+3\,x^4-6\,x^5-x^6$. 26. $32\,a^{15}-40\,a^{12}x+20\,a^9x^2-5\,a^6x^3+\frac{5}{8}\,a^3x^4-\frac{x^5}{32}$. 27. $x^6-15\,x^5+81\,x^4-185\,x^3+162\,x^2-60\,x+8$.

**Page 193.**—2. $m^3n^2x$. 4. $4\,n^5$. 6. $9\,a^2c^{12}$. 8. $8\,xy^3$. 10. $a(a+x)^2$. 11. $2(2+x)(3-x)^2$. 13. $n(x+a)^3(x-6)^4$. 15. $a^2(c+d)(x-y)^3$.

**Page 195.**—1. $x+4$. 2. $2\,a+3$. 3. $2\,x-7$. 4. $1+3\,m$. 6. $2\,ac-b$. 8. $7\,x+4\,y$. 10. $a^2-3\,a+2$. 11. $x^2-x-1$. 12. $x^2-x+1$.

**Page 196.**—1. 3, two. 3. 5, three. 5. 4, two. 7. 8, two. 9. 2, three. 11. 2, three. 12. 1, four.

**Page 197.**—1. 24. 2. 75. 3. 64. 5. 56. 7. 42. 9. 97.

**Page 198.**—1. 38. 2. 53. 3. 74. 5. 83. 7. 45. 9. 48. 11. 354. 12. 906. 13. 255. 14. 507. 15. .25. 16. .12. 17. .055. 18. $\frac{1}{7}$. 19. $\frac{9}{11}$. 20. 33.38+. 21. 1.259. 22. 1.442. 23. 1.709.

**Page 203.**—2. $x^{\frac{3}{5}}$. 4. $a^{\frac{1}{4}}+c^{\frac{3}{4}}$. 6. $y^{\frac{3}{5}}$. 8. $c^{\frac{1}{3}}+m^{\frac{1}{3}}$. 10. $\sqrt{a}\,\sqrt[3]{b}$. 11. $\frac{\sqrt[4]{c}}{\sqrt{x}}$. 12. $\sqrt[3]{a}-3\,\sqrt[4]{b}$. 15. $\sqrt[5]{\frac{n^3}{x^2}}$. 17. $\frac{1}{a^2}$. 19. $\frac{b}{c^3x^4}$. 20. $\frac{n}{m^2x^3}$. 22. $\frac{ac^2y^5}{b^2x^2}$. 24. $\frac{c^4x^5y^2}{a^2b^3z}$. 25. $a^2b^{-3}c^{-4}$. 27. $x^3y^{-1}z^2$. 29. $c^{-2}d^2n^{-5}$.

31. 243. 32. $-32$. 33. 2. 34. $\frac{1}{121}$. 36. $\frac{1}{3}$. 37. $\frac{1}{4}$. 39. $\frac{a^4d^6}{c^8}$. 41. $c^{\frac{1}{8}}$. 42. $\frac{1}{\sqrt{a^9c^4}}$. 43. $x^{\frac{11}{12}}$. 44. 12. 45. $a^{10}b^9$. 47. $c^{\frac{31}{24}}$. 48. $\sqrt[6]{\frac{r^4x^6}{y^3}}$. 49. 4. 50. 1. 52. $\frac{1}{3}$. 53. 4. 54. $\frac{1}{4}$.

**Page 204.** — 1. $a-b$. 2. $c+2\sqrt{cx}+x$. 3. $c-2\sqrt{cx}+x$. 5. $a^1+2a^2+a^3$. 6. $x^{\frac{17}{12}}-x^{\frac{7}{12}}$. 7. $a^{-5}-3a^{-3}b^2+a^{-2}b^2-2a^{-1}b^3$. 8. $2x^{-1}y-10xy^{-1}+8x^3y^{-3}$. 9. $a^{\frac{3}{2}}b^{-\frac{1}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{-\frac{1}{2}}b^{\frac{3}{2}}$. 10. $x^{\frac{5}{2}}+x^{\frac{3}{2}}y-xy^{\frac{3}{2}}-y^{\frac{5}{2}}$. 11. $a+a^{\frac{1}{2}}b+b^2$. 12. $6x^2-7x^{\frac{5}{3}}-19x^{\frac{4}{3}}+5x+9x^{\frac{2}{3}}-2x^{\frac{1}{3}}$. 13. $x^{-2}-3x^{-\frac{4}{3}}+3x^{-\frac{2}{3}}-1$. 14. The product of $x^{\frac{1}{2}}-y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ is the rational $x-y$.

**Page 205.** — 1. $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. 2. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$. 3. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$. 5. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}$. 6. $a^{\frac{4}{5}}-a^{\frac{3}{5}}b^{\frac{1}{5}}+a^{\frac{2}{5}}b^{\frac{2}{5}}-$, etc. 7. $x+y$. 8. $x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{\frac{1}{4}}+$, etc. 9. $a^{\frac{1}{3}}+b^{\frac{1}{3}}$. 10. $x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}$. 11. $2x^2-3xy+2y^2$.

1. $3x^{-2}-3x^{-1}y^{\frac{1}{2}}+y$. 2. $2a^{\frac{1}{2}}-3b^{\frac{1}{3}}+4c^{\frac{1}{4}}$.

**Page 206.** — 1. The area of a circle is equal to $\pi$ times the square of the radius. 2. $R=\sqrt{\frac{A}{\pi}}$; the radius of a circle is equal to the square root of the area divided by $\pi$. 4. $R=\sqrt{\frac{S}{4\pi}}$. 5. $R=\sqrt[3]{\frac{3V}{4\pi}}$. 6. The volume of a cylinder is equal to $\pi$ times the altitude multiplied by the square of the radius of the base. 9. 5 yd. 10. 1.5 in. 11. 2.5 in., nearly. 12. 8 ft.

**Page 208.** — 1. $\sqrt{a}$. 2. $\sqrt{ab}$. 3. $\sqrt{3c^2y}$. 4. $\sqrt[4]{a^3}$. 6. $\sqrt{3ac^3}$. 8. $\sqrt[3]{c^2x^2}$. 10. $\sqrt[3]{4}$. 12. $\sqrt{ab^3c^2}$. 13. $\sqrt{\frac{5}{4}}$. 14. $\sqrt{\frac{n}{2a^3}}$. 15. $\sqrt{\frac{2a}{x+2}}$.

**Page 209.** — 1. $2\sqrt{2}$. 2. $6\sqrt{5}$. 3. $5a\sqrt{ab}$. 4. $3\sqrt{3}$. 5. $4\sqrt[3]{2}$. 6. $3a^2b\sqrt{b}$. 8. $6\sqrt[4]{5}$. 10. $3\sqrt{5}$. 11. $24\sqrt{5}$. 12. $15x^2\sqrt{3y}$. 14. $25\sqrt[3]{4}$. 17. $24\sqrt[4]{2}$. 18. $2xy\sqrt[3]{20x^2}$. 20. $4\sqrt[6]{5}$. 22. $2\sqrt[5]{7}$. 24. $6a\sqrt[4]{5c}$. 26. $8a^3b\sqrt[5]{2c}$. 28. $30cx^2\sqrt[3]{2xy}$. 30. $2xy\sqrt[3]{2y-5x}$. 32. $(x-3)\sqrt{a}$. 34. $(x+y)\sqrt{x-y}$. 36. $4ab\sqrt[3]{3ab^2+5b}$.

**Page 210.** — 1. $\frac{1}{3}\sqrt{5}$. 2. $\frac{1}{8}\sqrt{14}$. 3. $\sqrt{6}$. 4. $\frac{1}{14}\sqrt{21}$. 6. $\frac{1}{15}\sqrt{165}$. 7. $2\sqrt{6}$. 8. $\frac{1}{4}\sqrt{6}$. 11. $2\sqrt{30}$. 12. $\frac{2}{3}\sqrt{15}$.

14. $\frac{1}{10}\sqrt[3]{300}$. 16. $\frac{2}{11}\sqrt[3]{196}$. 18. $\frac{1}{3}\sqrt[3]{20}$. 20. $\frac{1}{b}\sqrt{ab}$. 22. $\frac{1}{3}\sqrt[4]{108}$.
24. $\frac{1}{b}\sqrt[3]{a^2b}$. 25. $\frac{a}{b^2}\sqrt{bx}$. 26. $\frac{1}{b}\sqrt[3]{bc^2x}$. 27. $\sqrt[4]{abc}$. 28. $(a^2-b^2)^{\frac{3}{2}}$.
29. $x^3(x^4-1)^{\frac{3}{4}}$.

**Page 211.**—1. $\sqrt{20}$. 2. $\sqrt[4]{162}$. 3. $\sqrt{12\,a^3b}$. 6. $\sqrt[3]{a^7b^4}$.
9. $\sqrt[4]{32\,a^9b^4c}$. 11. $\sqrt[5]{a^{11}}$. 12. $\sqrt{18\,x}$. 14. $\sqrt[3]{c^9}$. 15. $\sqrt[3]{-54\,x}$.
16. $\sqrt{\frac{1}{5}}$. 17. $\sqrt[3]{\frac{1}{12}}$. 18. $\sqrt{\frac{a}{b}}$. 19. $\sqrt{\frac{1}{2}}$. 21. $\sqrt[3]{\frac{a^2}{b^2}}$. 22. $\sqrt{45\,a^5b^3c}$.
23. $\sqrt{4\,a^4x^2+12\,a^3x^3}$. 24. $\sqrt{\frac{x+y}{x-y}}$. 25. $\sqrt{x^2-1}$. 26. $\sqrt{\frac{x-y}{x+y}}$.
27. $\sqrt[3]{a-b}$.

**Page 212.**—1. $\sqrt[6]{125}$, $\sqrt[6]{121}$. 2 $\sqrt[12]{16}$, $\sqrt[12]{27}$, $\sqrt[12]{25}$. 3. $\sqrt[15]{32}$, $\sqrt[15]{27}$.
4. $\sqrt[10]{32}$, $\sqrt[10]{9}$, $\sqrt[10]{5}$. 6. $\sqrt[18]{9}$, $\sqrt[18]{64}$, $\sqrt[18]{343}$. 8. $\sqrt[4]{3}$. 9. $\sqrt{7}$. 10. $\sqrt[3]{5}$.
12. $\sqrt[5]{x^2}$. 14. $\sqrt[3]{32}$. 16. $\sqrt[4]{11}$. 17. $6\sqrt{5}$. 18. $6\sqrt{11}$. 19. $\sqrt[3]{\frac{14}{15}}$.

**Page 214.**—1. $7\sqrt{2}$. 2. 0. 3. $\sqrt{3}$. 4. $32\sqrt{2}$. 5. $4\,a\sqrt{c}$.
6. $11\sqrt[3]{5}$. 7. $2\sqrt[3]{3}+3\sqrt[3]{2}$. 8. $2\sqrt[4]{3}$. 10. $14\sqrt{2}+\sqrt{7}$. 12. $7\,ab\sqrt{2\,a}$.
13. $(2-a)\sqrt[4]{5}$ 14. $26\,a^2\sqrt[3]{3\,c}$. 15. $4\,a^2x^3\sqrt{3\,ay}$. 16. $a^2x\sqrt[3]{4\,x}$.
17. $\frac{1}{2}\sqrt{6}$. 18. $\frac{1}{4}\sqrt{6}$. 20. $\frac{1}{12}\sqrt[3]{18}$. 22. $\frac{4}{b}\sqrt{ab}$. 23. 0. 24. 0.
25. $\frac{7}{6}\sqrt{3}$. 26. $\frac{6}{5}\sqrt[3]{25}$. 28. $13\sqrt[3]{5}$. 29. $\frac{3}{5}\sqrt{15}$. 30. $\frac{10}{3}\sqrt[4]{3}$.

**Page 215.**—1. $3\sqrt{2}$. 2. $30\sqrt{2}$. 3. 4. 4. 4. 6. 60. 8. $2\sqrt{2}$.
9. $\sqrt[3]{50}$. 10. $\sqrt{10}$. 12. $\sqrt[4]{54}$. 13. $\sqrt[6]{24}$. 14. $\sqrt[6]{972}$. 15. $20\sqrt[12]{432}$.
17. $70\,ab\sqrt{2\,a}$. 19. $70\,a^3x\sqrt{2\,ac}$. 20. $\frac{9\,a}{2\,b}$.

**Page 216.**—22. $\frac{3}{11}\sqrt{3}$. 23. 1. 24. $4\sqrt[3]{15}$. 25. $\frac{1}{3}\sqrt[3]{36}$.
26. $12\sqrt[6]{432}$. 27. $\frac{1}{3}\sqrt[3]{72}$. 29. 2. 30. $14+8\sqrt{3}$. 31. $-22-7\sqrt{6}$.
32. $-39+7\sqrt{7}$. 33. $4+2\sqrt{10}$. 34. $56+12\sqrt{35}$. 35. $a+c-b+2\sqrt{ac}$.
36. $11\,x-9-3\sqrt{1-x^2}$. 37. $-78-47\,x+17\sqrt{6+13\,x+6\,x^2}$.
38. $20\,x-12+12\sqrt{1-x-6\,x^2}$.

**Page 218.**—1. $\frac{3}{2}\sqrt{2}$. 2. $\frac{5}{4}\sqrt{2}$. 3. $\frac{2\sqrt{3}+3}{3}$. 4. $\frac{1}{2}\sqrt[3]{2}$.
6. $\frac{1}{4}(2-\sqrt[3]{10})$. 8. $\sqrt[3]{4}$. 10. $\frac{19-8\sqrt{3}}{13}$. 11. $\frac{7+3\sqrt{5}}{2}$. 12. $5+2\sqrt{6}$.
13. $\frac{13+2\sqrt{30}}{7}$. 15. $\frac{23+4\sqrt{15}}{17}$. 16. $\frac{5+a+4\sqrt{a+1}}{3-a}$.
17. $2\,a^2-1-2\,a\sqrt{a^2-1}$. 19. $\frac{a+\sqrt{a^2-x^2}}{x}$. 20. $\sqrt{a^4-1}-a^2$.

**Page 219.** — 1. $16a^2$. 2. $96x^8\sqrt[3]{6}$. 3. $9\sqrt{c}$. 4. $-x^9\sqrt{a^{15}}$. 6. $-27a\sqrt[4]{a^2x^8}$. 8. $256a^4b^2\sqrt[3]{b^2}$. 10. $\sqrt[6]{2000}$. 12. $\sqrt{a}\sqrt[7]{b^2}$. 14. $3\sqrt[3]{2a^2}$. 16. $\frac{1}{2}\sqrt[3]{12}$. 18. $\sqrt[4]{a^3}$. 19. $3\sqrt{6}$. 21. $\sqrt[3]{a^2}$. 22. $2\sqrt{3}$. 24. $a\sqrt{a}$. 26. $b^{20}$. 26. $c^3\sqrt[5]{c^4}$. 31. $\sqrt[3]{4}$, $\sqrt[4]{2a}$. 32. 16, 125.

**Page 220.** — 1. $1+\sqrt{3}$. 2. $2-\sqrt{3}$. 3. $\sqrt{2}+\sqrt{5}$. 4. $2\sqrt{2}+\sqrt{10}$. 6. $\sqrt{10}-\sqrt{3}$. 8. $3-\sqrt{2}$. 9. $2\sqrt{5}-3\sqrt{2}$. 11. $5+\sqrt{10}$. 14. 7. 16. 6. 18. 9.

**Page 223.** — 1. $16\sqrt{-1}$. 2. $11+12\sqrt{-1}$. 3. $3\sqrt{-1}$. 4. $5+\sqrt{-2}$. 5. $-24$. 6. $6+17\sqrt{-1}$. 7. 5. 8. $\frac{8+15\sqrt{-1}}{17}$. 9. $-64\sqrt{-1}$. 10. 29. 12. 107. 14. $\frac{21+20\sqrt{-1}}{29}$. 16. $\frac{31-42\sqrt{-2}}{67}$. 18. $23+2\sqrt{-1}$. 19. $\frac{9+5\sqrt{-7}}{16}$. 21. $\frac{-7-9\sqrt{-2}}{11}$.

**Page 224.** — 1. 27. 2. 25. 3. $\pm\frac{8}{3}\sqrt{3}$. 4. 9. 5. 50. 6. $\pm\frac{4}{3}$. 7. $\pm 57\sqrt{57}$. 8. $\pm 13\frac{1}{2}$. 10. 16. 12. $\frac{4}{3}$. 14. $5\frac{1}{2}$. 16. 243. 18. $\frac{8}{27}$. 20. 32. 22. $\frac{1}{2}$ or 8. 23. $\pm 8$ or $\pm 27\sqrt{-1}$. 24. 1 or 27. 26. 1 or 16. 27. 1 or $\frac{1}{2}\sqrt[3]{18}$. 28. $\pm\frac{1}{3^3}$ or $\pm\frac{1}{3}\sqrt{2}$. 29. 2 or $\frac{1}{3^3}$.

**Page 226.** — 1. 3. 2. 5. 3. 9. 4. 4. 5. 4. 6. 4. 8. 81. 9. 12. 10. 5. 11. 4. 12. 0. 13. 0. 14. 2. 15. 2. 16. 25. 17. $56\frac{1}{4}$. 18. 6. 19. $\frac{9b^2}{a}$. 20. 6. 21. $\frac{1}{3}$. 22. $\pm\sqrt{3}$, 0.

**Page 227.** — 2. 2, $-10$. 4. 8, $-3$. 6. 6, $-\frac{1}{6}$. 8. $-2\frac{2}{3}$, $-3\frac{1}{3}$. 11. 10, $-9$. 13. 2, $-7$. 15. 6, 13. 16. 7, $-6\frac{1}{3}$. 17. 4, $-1\frac{1}{3}$. 18. 3, $3\frac{1}{3}$. 19. 9, $-8\frac{2}{7}$.

**Page 230.** — 1. $-1$. 2. 4, $-1\frac{1}{3}$. 3. 3. 4. 4, $-1\frac{2}{3}$. 5. $-9\frac{1}{2}$. 6. 9, $1\frac{2}{13}$. 7. 3. 8. 25. 9. 6, $3\frac{1}{13}$. 11. no root. 12. 1, $1\frac{1}{4}$. 13. 1, $4\frac{1}{3}$. 14. 0, 4. 15. 0, $\frac{2ab-ac-bc}{a+b-2c}$. 16. $\frac{2a}{a+b}$, $-\frac{2b}{a+b}$. 17. $\frac{p\sqrt{q}\pm q\sqrt{p}}{p-q}$. 18. $\frac{n}{n^2-m^2}(bn\pm\sqrt{a^2m^2+b^2m^2-a^2n^2})$.

**Page 231.** — 1. $6b$, $-2b$. 2. $-\frac{1}{3a}$, $\frac{3}{4a}$. 3. $7n$, $-2n$. 4. 1, $a$. 5. $2a$, $\frac{a}{2}$. 6. $-2a$, $-14$. 7. $\frac{c}{b}$, $-\frac{b}{c}$. 8. $\frac{b^2}{ac}$. 9. $\frac{2a+c}{2}$, $\frac{a+2c}{2}$. 11. $a-b$, $c$.

1. .41, $-2.41$. 2. 4.19, $-1.19$. 3. 2.73, $-.73$. 4. 3.85, $-2.85$. 5. 3.73, .27. 6. 2.39, .28.

**Page 232.** — **1.** $3, -4\frac{2}{3}$. **2.** $-8, -\frac{1}{3}$. **4.** $9, \frac{1}{2}$. **6.** $\frac{4}{5}, 1\frac{1}{2}$. **8.** $5, -5\frac{3}{4}$. **10.** $12\frac{1}{3}, 3\frac{1}{4}$. **12.** $24, -6$. **14.** $3, \frac{1}{2}$. **16.** $2, 8$. **17.** $n, -n-2r$. **18.** $3n, 4r-3n$.

**Page 233.** — **1.** Real, rational, unequal. **2.** The same. **3.** Same. **4.** Same. **5.** Real, rational, equal. **6.** Real, rational, unequal. **7.** Imaginary. **8.** Same. **9.** Real, irrational, unequal. **10.** Same. **11.** Same. **12.** Same. **13.** 1. **14.** 1. **15.** 3. **16.** 7.

**Page 234.** — **2.** $x^2-2ax+a^2-b^2$. **4.** $x^2-2bx+b^2-a^2$. **6.** $x^2-6cx+9c^2-1=0$. **7.** $4x^2+48x+143=0$. **8.** $x^2-6x+7=0$. **9.** $abx^2+(b^2-a^2)x-ab=0$. **10.** $4x^2-4ax+a^2-b^2=0$. **11.** $12x^2-ax-6a^2=0$. **12.** $4x^2-4ax+a^2+b^2=0$. **13.** $x^2-ax-b=0$.

**Page 237.** — **1.** $\pm 2, \pm\sqrt{10}$. **2.** $1, -2$. **3.** $9, 41\sqrt{-41}$. **4.** $16, \frac{14}{5}\frac{4}{2}\frac{4}{5}\frac{1}{1}$. **5.** $1, \frac{7}{5}\sqrt[3]{63}$. **6.** $\frac{1}{4}, \frac{1}{3}$. **7.** $\frac{1}{4}, 4$. **8.** $8, \frac{125}{64}$. **9.** $1$. **10.** $4, -9$. **11.** $0, -5$. **12.** $1, -4, \frac{-3\pm\sqrt{109}}{2}$. **13.** $-1\pm\sqrt{3}, -1\pm\sqrt{2}$. **14.** $2, -2, 3, 7$. **15.** $3, \frac{1}{3}, \frac{1}{3}(5\pm\sqrt{28})$. **16.** $1, -1, 5, 7$. **17.** $4, 1, 3, 2$. **18.** $-1, -2, \frac{-3\pm\sqrt{33}}{2}$. **19.** $\frac{-3\pm\sqrt{29}}{4}, \frac{-3\pm\sqrt{-7}}{4}$.

**Page 238.** — **1.** $2, -1\pm\sqrt{3}$. **2.** $1, 3\pm\sqrt{6}$. **3.** $-1, \frac{1}{2}\pm\frac{1}{2}\sqrt{-31}$. **4.** $-1, 3, 4$. **5.** $0, 2, 3$. **6.** $-1, \frac{1}{10}(3\pm\sqrt{-131})$. **7.** $-3, \frac{3}{2}\pm\frac{1}{2}\sqrt{-7}$. **8.** $1, 2, 3$. **9.** $0, \frac{1}{2}, -\frac{2}{3}$. **10.** $0, 1, 1, 1$. **11.** $-2, \frac{2}{3}, -\frac{2}{3}$. **12.** $6, 8, -2$. **13.** $2, 3.2, -2$. **14.** $0, -5, -3\pm\sqrt{26}$. **15.** $2, 1, 3\frac{1}{2}$. **16.** $4, \pm\sqrt{-5}$. **17.** $0, 1\frac{2}{5}, \pm\sqrt{-3}$. (See § 90.)

**Page 239.** — **2.** 12 ft. **3.** 12, 16, 20 in. **4.** 8, 15, 17 in. **5.** 39, 52 in. **6.** 9, 12, 15 ft. **7.** 39, 80, 89 in.

**Page 240.** — **1.** 10, 11. **2.** 14, 15. **3.** 6, 7, 8, 9. **4.** 14, 16 rd. **5.** 7, 14 ch. **6.** 28, 45, 53 rd. **7.** 27, 36 mi. **8.** 6, 8 hr. **9.** 76 mi.

NOTE. In the answers to problems on pages 241 to 245 all the pairs of values for $x$ and $y$ are, in general, not given.

**Page 241.** — **1.** $x=3, y=2$. **2.** $x=4, y=3$. **3.** $x=8, y=2$. **4.** $x=15, y=10$. **5.** $x=4, y=3$. **6.** $x=7, y=3$. **7.** $x=5$ or $6, y=4$ or $3\frac{1}{3}$. **8.** $x=3, y=-4$. **9.** $x=8$ or $17\frac{7}{8}, y=6$ or $-13\frac{1}{3}$. **10.** $x=3, y=1$. **11.** $x=8, y=4$. **12.** $x=4$ or $-\frac{1}{2}, y=0$ or $-4\frac{1}{2}$.

**Page 242.** — **1.** $x=5$ or $-6, y=5$ or $-6$. **2.** $x=2$ or $-\frac{3}{4}, y=1$ or $-\frac{3}{5}$. **3.** $x=1, -\frac{1}{3}, \frac{-32\pm 8\sqrt{21}}{5}, y=-1, +\frac{1}{3}, \frac{-8\pm 2\sqrt{21}}{5}$. **4.** $x=2, -3, -2, \frac{3}{4}, y=2, -3, -4, \frac{3}{2}$. **5.** $x=0, -\frac{1}{3}, 2, y=-\frac{1}{3}, -\frac{1}{3}, 2$. **6.** $x=3\frac{5}{9}, -3, y=1\frac{5}{27}, -1$. **7.** $x=2\pm 2\sqrt{5}, 3, -\frac{2}{3}, y=\frac{1}{2}(1\pm 5), 1, -\frac{2}{3}$. **8.** $x=5, -30, y=1, -6$.

**Page 243.** — **1.** $x=\pm 3, \pm\frac{1}{2}\sqrt{2}, y=\pm 2, \pm\frac{1}{2}\sqrt{2}$. **2.** $x=\pm 2, \pm\frac{1}{5}\sqrt{10}, y=\pm\frac{1}{2}, \mp\frac{2}{5}\sqrt{10}$. **3.** $x=4, \sqrt{2}, y=2, -3\sqrt{2}$. **4.** $x=2, \frac{5\sqrt{31}}{31}, y=3, -\frac{6\sqrt{31}}{31}$. **5.** $x=\pm 5, y=\pm 2$. **6.** $x=\pm 2, \pm 4\sqrt{3}, y=\pm 5, \pm\sqrt{3}$. **7.** $x=\pm 6, \pm 4\sqrt{-2}, y=\pm 4, \mp 3\sqrt{-2}$. **8.** $x=\pm 2, \pm\sqrt{2}, y=\pm 4, \mp 3\sqrt{2}$. **9.** $x=\pm 2, \pm 2\sqrt{3}, y=\pm 3, \mp 3\sqrt{3}$. **10.** $x=\pm 3, \pm\frac{5}{3}, y=\pm 5, \pm\frac{15}{8}$.

**Page 245.** — **1.** $x=2, y=5$. **2.** $x=3, y=4$. **3.** $x=8, y=6$. **4.** $x=5, y=3$. **5.** $x=15, y=10$. **6.** $x=8, y=2$. **7.** $x=3, y=2$. **8.** $x=\frac{1}{3}, y=\frac{1}{3}$. **9.** $x=2, y=1$. **10.** $x=3, -3+\sqrt{3}, y=2, -3-\sqrt{3}$. **11.** $x=3, y=2$. **12.** $x=5, y=4$. **13.** $x=10, y=3$. **14.** $x=5, y=4$. **15.** $x=2, y=1$. **16.** $x=5, y=4$. **17.** $x=5, y=3$. **18.** $x=3, \frac{1}{2}(\sqrt{-31}+1), y=2, \frac{1}{2}(\sqrt{-31}-1)$.

**Page 246.** — **1.** 7, 4. **2.** 4, 2. **3.** $\pm 7$ and $\pm 5$, or $\pm 6\sqrt{2}$ and $\pm\sqrt{2}$. **4.** 6 and 4, or $-4$ and $-6$. **5.** 24 ch., 5 ch. **6.** 9 mi., 3 mi. **7.** 95. **8.** 8 in., 15 in. **9.** 6 mi., 4 mi.

**Page 247.** — **10.** 50 ¢, 25 ¢. **11.** 20 da., 30 da. **12.** 4 yd., 5 yd. **13.** 5 mi., 2 mi. per hour. **14.** 48 ft. **15.** $\frac{1}{2}\sqrt{5}, \frac{1}{4}(5+\sqrt{5})$. **16.** 20 ¢.

**Page 248.** — **17.** $\frac{1}{2}(s+\sqrt{s^2-4p}), \frac{1}{2}(s-\sqrt{s^2-4p})$. **18.** 13, $-4$. **19.** $\frac{1}{2}(\sqrt{4a+l^2}+l), \frac{1}{2}(\sqrt{4a+l^2}-l)$. **20.** 8 in., 15 in. **21.** $\frac{1}{2}(\sqrt{d^2+2a}+\sqrt{d^2-2a}), \frac{1}{2}(\sqrt{d^2+2a}-\sqrt{d^2-2a}$. **22.** 56 in., 33 in. **23.** $\sqrt{\frac{dp}{l}+\frac{p^2}{4}}-\frac{p}{2}$. **24.** 30. **25.** $\sqrt{\frac{bd}{a}+\frac{d^2}{4}}-\frac{d}{2}$. **26.** $12\frac{1}{2}$ mi.

**Page 254.** — **2.** 25. **3.** 24. **4.** 77. **5.** $2\frac{2}{11}$. **6.** 30. **7.** $5\frac{2}{5}$. **8.** $\frac{1}{7}$. **9.** 9. **10.** 13. **11.** 20. **12.** 12. **13.** 15. **14.** $6\frac{2}{3}$. **16.** $cd$. **18.** $8\frac{1}{3}$. **20.** 8. **22.** $\frac{162\,b^3}{49}$. **23.** $9\frac{1}{18}$. **25.** $\frac{(a-b)^3}{(a+b)^3}$. **26.** 20. **28.** $ab\sqrt{-1}$. **29.** $3\frac{3}{5}$. **31.** $6\frac{2}{3}$. **32.** $x=4, y=7$. **33.** $x=8, y=11$. **34.** $x=5, y=6$. **35.** $x=\frac{3p(3c+2d)+2b(4n+3q)}{8bm-9ap}$, $y=\frac{4m(3c+2d)+3a(4n+3q)}{8bm-9ap}$.

**Page 255.** — **1.** 21, 28. **2.** 7. **3.** 48 yr., 20 yr. **4.** 3, 5. **5.** By comp. and div. we have $2(x^2+1):2x=4x:2$, etc. **6.** 18, 27. **7.** 14, 18. **8.** 15 yd., 25 yd. **9.** 6, 9. **10.** $5\frac{13}{16}, 4\frac{7}{16}$.

**Page 256.** — **2.** 720 cu. in. **3.** 6, 9, 12 in., 468 sq. in. **4.** $S=2ab+2ac+2bc$. **5.** 3, 6, 8 in. **6.** 5, 8, 12 in. **7.** 3 in. square, or 9 sq. in. **8.** $\sqrt{BSE}$. **9.** 30, 20, 15 in.

**Page 260.** — **1.** 96. **2.** 5. **3.** 140. **4.** 5. **5.** 12. **6.** $z = 3(x + y)$. **7.** $y = \sqrt[3]{\frac{x^2}{2}}$. **8.** 512. **9.** 16.

**Page 261.** — **12.** 1608½ ft. **13.** 33⅓ cu. in. **14.** 180. **15.** 5000 lb. **16.** 12½.

**Page 262.** — **2.** $s$ varies as the square of the time. **3.** 257.28 ft. **4.** 402 ft. **5.** $t = \frac{v}{g}$. **6.** 31.1 sec. **7.** $t = \sqrt{\frac{2s}{g}}$. **8.** 7.9 sec. **9.** 144.72 ft. **10.** $s = \frac{v^2}{2g}$. **11.** 3886.8 ft.

**Page 264.** — **1.** 2, 7, 12, ··· 47. **2.** 6, 12, 20, ··· 132. **4.** 0, 0, 2, ··· 72. **6.** 12, 20, 30, ··· 156. **8.** 4, 0, −2, ··· 40. **10.** 1, 4, 10, ··· 220. **12.** $\frac{1}{8}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \cdots \frac{168}{143}$. **14.** −2, 1, 7, ··· 1531.

**1.** $n(n+2)$. **2.** $n(n-1)$. **3.** $n+5$. **4.** $3n+2$. **5.** $\frac{n}{n+1}$. **6.** $\frac{n+5}{n+2}$. **7.** $(n+2)^2$. **8.** $3 \times 2^{n-1}$.

**Page 265.** — **3.** 149. **4.** 5. **5.** 21½. **6.** −11. **7.** 51. **8.** 18. **9.** $a = 4$, $d = -3$. **10.** $a = 1\frac{2}{3}$, $d = \frac{5}{6}$. **11.** $\frac{37 - 7n}{4}$.

**Page 266.** — **12.** $d = \frac{l - a}{n - 1}$. **13.** 2. **14.** $a = l - (n-1)d$. **15.** 7. **16.** $n = \frac{l - a}{d} + 1$. **17.** 16. **18.** 99. **20.** 11. **21.** $an = 2n - 1$; the A. P. is 1, 3, 5, ···. **22.** $a = p + q$, $d = q$.

**1.** 4, [13, 22, 31, 40, 49], 58. **2.** −5, [−2, 1, ··· 13, 16], 19. **3.** 3, $[3\frac{1}{10}, 3\frac{1}{5}, \cdots 3\frac{4}{5}, 3\frac{9}{10}]$, 4. **4.** $4\frac{1}{8}$, $[4\frac{1}{4}, 4\frac{1}{8}, \cdots 5\frac{1}{8}, 5\frac{5}{12}]$, $5\frac{1}{2}$.

**Page 267.** — **2.** 558. **3.** 830. **4.** 3999. **5.** −375. **6.** 165⅓. **7.** 5050. **8.** $n^2$. **9.** 97½. **12.** $1980x$. **13.** $6605m$.

**Page 268.** — **15.** $\frac{193n^2}{12}$ ft., 1608⅓ ft.

**1.** 13. **2.** 13. **3.** 528. **4.** 7. **5.** $a = 11$, $n = 7$. **6.** 19.

**Page 269.** — **1.** 105. **2.** 3. **3.** $3n - 2$; 1, 4, 7, 9. **4.** 1, 3, 5, 7, ···. **5.** $a = 2$, $n = 11$.

**Page 270.** — **6.** 5, 7, 9, 11. **7.** 8, 10, 12. **8.** 2, 3, 4. **9.** 5, 6, 7, 8. **10.** 2, 3, 4, 5, 6, 7, 8. **11.** 5. **12.** 3, 8, 13, ···. **13.** 3 or 10 da., 36 or 120 mi. **14.** 19 mi. 160 rd. 640 ft.

**Page 271.** — **3.** 327,680. **4.** $\frac{8}{9}$. **5.** 2560. **6.** $\frac{5}{3}$. **7.** $\frac{7}{4}$. **8.** $\frac{4}{3}$. **9.** $\frac{1}{256}$. **10.** $\frac{896}{729}$. **11.** $a = 80$, $r = 2$. **12.** $3 \times 5^{n-1}$.

**Page 272.**—**1.** 2. **2.** $\frac{8}{3}$.

**1.** 9. **3.** $-1{,}171{,}875$. **4.** $a_n = \frac{8}{3} \times (\frac{1}{2})^{n-1}$; $\frac{8}{3}$, $\frac{4}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{10}$. **5.** $a = \frac{5}{3}$, $r = 9$.

**Page 273.**—**1.** 4, [6], 9. **2.** $6\frac{1}{4}$, [$5\frac{5}{8}$], $5\frac{1}{2}$. **3.** 5, [20, 80], 320. **4.** 6, [18, 54, 162], 486. **5.** 2, [6, $\cdots$ 4374], 13, 122. **6.** 1, [$(-1)^{\frac{1}{4}}$, $(-1)^{\frac{1}{2}}$, $(-1)^{\frac{3}{4}}$], $-1$. **7.** $\frac{16}{27}$, [$\frac{8}{9}$, $\frac{4}{3}$, 2, 3, $\frac{9}{2}$], $6\frac{3}{4}$.

**1.** 2047. **2.** 9841. **3.** $3249\frac{7}{8}$. **4.** $\frac{1}{5}^2$ [$(\frac{2}{3})^6 - 1$]. **5.** $-\frac{2}{3}(2^{10} - 1)$. **6.** $9[1 - (\frac{2}{3})^n]$. **7.** $(\frac{2}{3})^{n-1} - \frac{2}{3}$.

**Page 274.**—**1.** $1\frac{1}{3}$. **2.** $4\frac{6}{11}$. **3.** 2. **4.** $\frac{27}{28}$. **5.** $\frac{1}{3}$. **6.** $\frac{1}{3}$. **7.** $\frac{1}{4}$. **8.** $\frac{n}{n-1}$. **9.** $\frac{n}{n+1}$. **10.** $\frac{25}{33}$. **11.** $\frac{28}{33}$. **12.** $\frac{365}{999}$. **13.** $\frac{28}{55}$. **14.** $\frac{1418}{3333}$.

**Page 275.**—**1.** $r = 3$, $n = 9$. **2.** $n = 9$, $s = 1022$. **3.** 20. **4.** 3, 4, 5.

**Page 276.**—**5.** 3. **6.** $r = 5$, $s = 15624$. **7.** 2. **8.** 5. **9.** $n = 9$, $l = 6561$. **10.** $\frac{3}{8}$, $\frac{9}{32}$. **11.** 32. **12.** 400 mi. **13.** \$21, \$42, \$84. **14.** 9, 27, 81, $\cdots$. **15.** £3 4 *s*.

**Page 277.**—**1.** $1\frac{1}{3}$. **2.** $-\frac{40}{9}$. **3.** $\frac{3}{11}$. **4.** $-1\frac{1}{12}$. **5.** 3, [$2\frac{4}{7}$, $2\frac{1}{4}$], 2. **6.** 3, [7, $-21$, $-4\frac{1}{5}$, $-2\frac{1}{3}$, $-1\frac{8}{13}$, $-1\frac{4}{17}$], $-1$. **7.** $-1$, [$-1\frac{1}{2}$, $-3$, $\infty$, 3, $1\frac{1}{2}$], 1. **8.** 2, 8. **9.** 40, 10.

**Page 292.**—**1.** 5.8. **2.** 1.1. **4.** 1.3. **6.** 11.2. **8.** 7. **10.** 6.2. **12.** 5.9. **14.** 11.

**Page 294.**—**2.** 16.6 lb. corn, 9.6 lb. hay. **3.** 5.8 oz. butter, 4.8 lb. bread. **6.** 16.2 lb. oats, 7.8 lb. fodder, 8.3 lb. hay. **8.** About 31 oz. milk, 33 oz. light bread.

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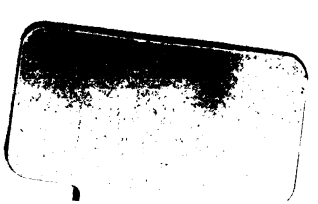
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